Ex. set # 4

#1. Let $\mathcal{H} = L^2(0,1)$ and Y denotes the subspace of constant functions. Compute the orthogonal projection $P_Y f$ for any $f \in \mathcal{H}$.

#2. Given a convex closed set $S \subset \mathcal{H}$ and a vector $x \in \mathcal{H}$, $x \notin S$ prove that there exists a unique $y \in S$ such that $\|x - y\| = \inf_{z \in S} \|x - z\|$.

#3 Prove the Han-Banach theorem for Hilbert space:

Let $\mathcal{H}$ be a Hilbert space. Given a closed subspace $Y \subset \mathcal{H}$ and a functional $\psi \in Y^*$ there exists $\Psi \in X^*$ such that $\Psi|_Y = \psi$ and $\|\Psi\|_{X^*} = \|\psi\|_{Y^*}$.

#4 Prove that any orthogonal system is linearly independent.

#5. Let $\mathcal{H}$ be a Hilbert space, $x, y \in \mathcal{H}$, $x, y \neq 0$. Prove that $\|x + y\| = \|x\| + \|y\|$ only if $x = cy$ with $c > 0$.

#6. Let $\mathcal{H}$ be a Hilbert space, $B_{\mathcal{H}}$ denotes the unit ball in $\mathcal{H}$: $B_{\mathcal{H}} = \{x \in \mathcal{H}, \|x\| \le 1\}$ and $\{x_n\}, \{y_n\} \subset B_{\mathcal{H}}$. Prove that if $\lim_{n\to\infty} \langle x_n, y_n \rangle = 1$ than $\|x_n - y_n\| \to 0$, $n \to \infty$.

#7. Let $\mathcal{H}$ be a Hilbert space, V - closed linear subspace. Prove that $(V^\perp)^\perp = V$.