Problem set 3

1. Given a Banach space prove that its norm is generated by somme inner product if and only if the parallelogram relation holds. (see hints in 1.4.24, Vershynin).

2. (Vershynin, ex. 1.4.22). Given two Hilbert spaces X and Y (with inner products and norms $\langle \cdot , \cdot \rangle_X$, $\|\cdot\|_X$ and $\langle \cdot , \cdot \rangle_Y$, $\| \ \|_Y$ respectively consider their direct sum

$X \oplus Y = \{ (x,y), x \in X, y \in Y \}$ provided with the inner product $\langle (x_1, y_1), (x_2, y_2) \rangle = \langle x_1, x_2 \rangle_X + \langle y_1, y_2 \rangle_Y$

Prove that $X \oplus Y$ also is a Hilbert space and write explicit expression for norm in $X \oplus Y$.

3. (Tchebyshev polynomials).

For $t \in (-1,1)$ set $T_n(t) = \cos(n \arccos t)$

Prove that:

- $T_n(t)$ is a polynomial of degree n
- They meet the recurrence relation

$$T_{n+1}(t) = 2t\,T_n(t) - T_{n-1}(t).$$

- For $n \geq 1$ the leading coefficient of $T_n(t)$ is 2^{n-1}, i.e. $T_n(t) = 2^{n-1} t^n + \dots$

- $\{T_n(t)\}$ are orthogonal in $L^2\left([-1,1], \frac{dt}{\sqrt{1-t^2}}\right)$,

more precisely
$$\int_{-1}^{1} T_n(t)\, T_m(t) \frac{dt}{\sqrt{1-t^2}} = \begin{cases} 0, & m \neq n \\ \pi, & n = m = 0 \\ \frac{\pi}{2}, & n = m \neq 0. \end{cases}$$

- (additional for fun) T_n satisfy the Tchebyshev differential equation

$$(1-t^2)\,T_n''(t) - t\,T_n'(t) + n^2\,T_n(t) = 0.$$