1. Consider the operator of integration in $C[0,1]$:

$$A : x \mapsto \int_0^t x(\tau)\,d\tau$$

Find (some) invariant subspaces of $A$

2. Let $X$ be a Banach space, $A \in \mathcal{L}(X,X)$ and $\lambda, \mu \in \mathbb{C}$ do not belong to spectrum of $A$.
Prove the resolvent relation:

$$R(A,\lambda) - R(A,\mu) = (\lambda-\mu) R(A,\lambda) R(A,\mu)$$

3. a) Find the polar representation for the operator $A : \mathbb{C}^2 \to \mathbb{C}^2$, generated by the matrix $\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$.

b). Give an example of $2\times 2$ matrix $A$, whose spectrum is $\{1\}$ but $A \neq A^*$

c) Give an example of $2\times 2$ matrix $A$ whose spectrum is $\{1\}$ and $A = A^*$

4. Let $\mathcal{H}$ be a Hilbert space. Given an operator $T \in \mathcal{L}(\mathcal{H})$ we define $e^T$ as

$$e^T = \sum_{k=0}^{\infty} \frac{1}{k!} T^k .$$

Prove that if $A$ is a selfadjoint operator that $e^{iA}$ is a unitary operator.

5. Let $A : L^2(0,1) \to L^2(0,1)$ be a Hilbert-Schmidt operator defined as

$$A : x \mapsto (Ax)(t) = \int_0^1 x(\tau)\, k(t,\tau)\, d\tau$$

with $\displaystyle\int_0^1 \int_0^1 |k(t,\tau)|^2 \, dt\, d\tau < \infty.$

Prove that $A$ is a compact operator.