Norwegian University of Science and Technology
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TMA4230 Functional Analysis
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Exercise set 6

1 Show that the following operator is closed.

a) On the space of continuously differentiable functions $C^1[0,1]$ on $[0,1]$ we define the differentiation operator $Tf(x) = f'(x)$.

Solution. We need to show that if $f_n \to f$ and $f'_n \to g$, then $f' = g$. The convergence in question is uniform, i.e. in the supremum-norm $\|\cdot\|_\infty$. We would like to use the fundamental theorem of calculus to prove the statement, so consider the following calculation for fixed $x \in [0,1]$:

$$\begin{aligned}\lim_{n\to\infty} f_n(x) - f_n(0) &= \lim_{n\to\infty} \int_0^x f'_n(t)\ dt \\ &= \int_0^x \lim_{n\to\infty} f'_n(t)\ dt \\ &= \int_0^x g(t)\ dt.\end{aligned}$$

This calculation deserves some explanation. The first step is just the fundamental theorem of calculus. In the second step, we have taken the limit inside the integral. Depending on your knowledge of measure theory, this could induce varying degrees of discomfort, as this is not always allowed. In this case it *is* allowed, since the convergence of $f'_n \to g$ is *uniform*[1].

The calculation shows that the functions $f_n(x) - f_n(0)$ converge to $\int_0^x g(t)\ dt$ pointwise. But we also know that $f_n(x) - f_n(0) \to f(x) - f(0)$! Thus, by uniqueness of limits, $f(x) = f(0) + \int_0^x g(t)\ dt$. To finish of the proof, we note that this implies that $f' = g$, which is what we wanted to prove.

2 Let X be a Banach space and T a bounded linear operator on X.

a) Show that the following are equivalent:

- T is invertible.
- The range of T is dense in X and there exists a $C > 0$ such that $\|Tx\| \geq C\|x\|$ for all $x \in X$.

[1] This is also the reason why a Taylor series can be integrated termwise – the series expansion converges uniformly.

Solution. First assume that T is invertible. Then T is surjective, so the range of T is dense. I claim that we can pick $C = \frac{1}{\|T\|}$ to get $\|Tx\| \geq C\|x\|$ (the case where $\|T\| = 0$ is both boring and trivial, so we can safely divide by $\|T\|$). This follows from the calculation

$$\|x\| = \|T^{-1}Tx\| \leq \|T\|\|Tx\|.$$

Then assume that the second part is true. We need to show that T is invertible, and clearly this would follow if we can show that T is bijective (the condition $\|Tx\| \geq C\|x\|$ shows that the inverse, if it exists, is bounded). T is clearly injective; if $Tx = 0$, then $0 = \|Tx\| \geq C\|x\|$, hence $x = 0$. Now let $y \in X$ – we need to find $x \in X$ such that $Tx = y$ to show that T is surjective. Since the range of T is dense in X, we can pick a sequence $(x_n)_{n\in\mathbb{N}}$ such that $Tx_n \to y$. In particular the sequence $(Tx_n)_{n\in\mathbb{N}}$ is Cauchy, and we will now show that this implies that $(x_n)_{n\in\mathbb{N}}$ is Cauchy and hence convergent. For every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $\|Tx_m - Tx_n\| < \epsilon$ whenever $m, n \geq N$. But we know that $\|x_m - x_n\| \leq \frac{1}{C}\|Tx_m - Tx_n\| < \frac{\epsilon}{C}$, hence $(x_n)_{n\in\mathbb{N}}$ is Cauchy and converges to some point x. Since $Tx_n \to y$ by assumption and $Tx_n \to Tx$ by what we just showed, we must have $Tx = y$ by uniqueness of limits.

3 Let T_a be the multiplication operator on ℓ^2 defined by a bounded sequence $a = (a_n)$.

a) If $\inf\{|a_n| : n = 1, 2, ...\} > 0$, show that T_a is invertible on ℓ^2 and find its inverse.

b) Show that T_a is not invertible on ℓ^2 for $a = (1/n)$.

c) If λ is not in the closure of $\{a_n : n = 1, 2, ...\}$, then show that $T_a - \lambda I$ is invertible on ℓ^2.

Solution. **a)** We will use the previous exercise. It is easy to check that $\|T_a x\| \geq C\|x\|$ for any $x \in \ell^2$, where $\sqrt{C} = \inf\{|a_n| : n = 1, 2, ...\}$. Furthermore, the range of T_a is dense: if we want to approximate $x = (x_n)_{n\in\mathbb{N}} \in \ell^2$ with an element in the range of T_a, just consider $T_a y_k$ where $y_k = (y_k)_n$ is given by

$$(y_k)_n = \begin{cases} \frac{x_n}{a_n} & \text{if } k \leq n \\ 0 & \text{otherwise} \end{cases}$$

(Just to be clear, $(y_k)_n$ is the n'th term of the sequence y_k). Then $T_a y_k = (x_1, x_2, ..., x_n, 0, 0, 0...)$, and clearly we can approximate x with such elements.

It should not come as a great surprise that T_a^{-1} is $T_{a'}$, where $a' = (1/a_1, 1/a_2, ...)$. The only non-trivial part is to check that $a' \in \ell^\infty$, which follows from the assumption $\inf\{|a_n| : n = 1, 2, ...\} > 0$.

b) To be invertible, T_a must be onto. We'll find an element in ℓ^2 with no preimage under T_a. Consider $a = (1/n)_n \in \ell^2$. If there exists $(x_n)_n \in \ell^2$ such that $T_a(x_n)_n = a$, then clearly $(x_n)_n = (1)_n \notin \ell^2$ (you get the equation $\frac{1}{n}x_n = \frac{1}{n}$ in every coordinate). Hence a does not have a preimage in ℓ^2, and T_a is not onto.

c) This is an application of **a)**. Note that $T_a - \lambda I$ is an operator of the same form as T_a: it is the operator $T_{a-\lambda}$. $T_{a-\lambda}$ is invertible by **a)**, since "λ not in the closure of $\{a_n : n = 1, 2, ...\}$" means exactly that $\inf\{|a_n - \lambda| : n = 1, 2, ...\} > 0$.

4 Suppose T is an open mapping between Banach spaces X and Y. Show that the preimage of a

dense subset in Y is dense in X.

Solution. Assume that $D \subset Y$ is dense, and let $U \subset X$ be an open set. We need to show that $T^{-1}(D) \cap U$ is non-empty. Since T is open, $T(U)$ is open, and since D is dense there must exist a $y \in T(U) \cap D$. Hence $y = Tx$ for some $x \in U$ (since $y \in T(U)$), and clearly $x \in U \cap T^{-1}(D)$.

5 Let X be a Banach space and T a (bounded) linear operator on X with domain $D(T)$.

Show that if there exists a sequence $(x_n) \in D(T)$ such that $\|x_n\| = 1$ and $Tx_n \to 0$,

then T does not have a bounded inverse.

Solution. Assume that T^{-1} is an inverse of T. Then we calculate that

$$\|x_n\| = \|T^{-1}T(x_n)\| \leq \|T^{-1}\|\|T(x_n)\|.$$

But we assume that the right hand side converges to 0, while the left hand side is constant. This is clearly a contradiction.