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Exercise set 1 – Solutions

1 Let $c = (c_n)_n$ a sequence of scalars. We denote by $(\delta_n)_n$ the standard basis. Show the following assertions.

a) If $1 \le p < \infty$, then $\sum_n c_n \delta_n$ converges in ℓ^p if and only if $c \in \ell^p$.

b) The series $\sum_n c_n \delta_n$ converges in ℓ^∞ if and only if $c \in c_0$.

Solution. **a)** We start by making clear what it means for a sum to converge in a Banach space X. If $a_n \in X$ for every $n \in \mathbb{N}$, the sum $\sum_{n=1}^{\infty} a_n$ converges in X if the sequence of partial sums $S_N = \sum_{n=1}^{N} a_n$ converges - this is simply the definition of convergence of a series. To check the convergence of a sequence S_N in a Banach space, it is sufficient to check that it is Cauchy. We therefore need to consider expressions of the form $\|S_N - S_M\|_X$ where $N, M \in \mathbb{N}$. The sequence $(S_N)_N$ is Cauchy if for every $\epsilon > 0$ we can find $N_\epsilon \in \mathbb{N}$ such that $\|S_N - S_M\|_X < \epsilon$ whenever $M, N \ge N_\epsilon$. More informally, we can make $\|S_N - S_M\|_X$ as small as we like by picking M, N large enough.

In this exercise we have two sums, namely $\sum_{n=1}^{\infty} c_n \delta_n$ and $\sum_{n=1}^{\infty} |c_n|^p$. We want to write down the expressions for $\|S_N - S_M\|_X$ in both cases, and compare them. The first sum is in ℓ^p, so if we denote its partial sums by S_N we find (convince yourself of this!) that

$$\|S_M - S_N\|_p = \left(\sum_{n=N+1}^{M} |c_n|^p \right)^{\frac{1}{p}}. \tag{1}$$

The second sum is in $\mathbb{R}$, and if we denote its partial sums by S'_N we find that

$$|S'_M - S'_N| = \sum_{n=N+1}^{M} |c_n|^p. \tag{2}$$

It is clear that we can make the expression in equation **??** as small as we like by picking M, N as large as we like *exactly* when we can do the same for the expression in equation **??**. After all, the only difference is a p'th root, and you should convince yourself that you can make the p'th root of a number small if and only if you can make the number itself small, even though the N_ϵ needed in the two different cases may differ. In other words, one of the sums is Cauchy exactly when the other is, so the statement is true.

b) We will take it as obvious that if $\sum_{n=1}^{\infty} c_n\delta_n$ converges in ℓ^∞, it must be to the sequence $(c_n)_n$. Let us write down what it means that $(c_n)_n$ converges to 0: for every $\epsilon > 0$ there must exist an $N_\epsilon \in \mathbb{N}$ such that $|c_n| < \epsilon$ whenever $n \geq N_\epsilon$. An equivalent way of writing this is that for every $\epsilon > 0$, there exists an $N_\epsilon \in \mathbb{N}$ such that $\sup_{n\geq N} |c_n| < \epsilon$ (think about this!). Let us refer to this equivalent convergence condition as †.

On the other hand: what does it mean that $\sum_{n=1}^{\infty} c_n\delta_n = (c_n)_n$ in ℓ^∞? Well, we need that for every $\epsilon > 0$ there is an $N_\epsilon \in \mathbb{N}$ such that $\|(c_n)_n - S_N\|_\infty < \epsilon$ whenever $N \geq N_\epsilon$, where S_N are the partial sums and $(c_n)_n$ refers to *the whole sequence* as an element of ℓ^∞. It is not difficult to find that

$$\|(c_n) - S_N\|_\infty = \sup_{n\geq N+1} |c_n|. \tag{3}$$

In other words, the definition that $S_N \to (c_n)$ in ℓ^∞ becomes more or less exactly the same condition as †! It should therefore be clear that $c_n \to 0$ if and only if the sum converges in ℓ^∞.

You should note that this means that the standard basis is *not* a Schauder basis for ℓ^∞. After all, there are many elements in ℓ^∞ that do not converge to zero, and we have just shown that their "basis expansions" do not converge. In later exercises we will see that ℓ^∞ is not separable, and that a non-separable space cannot have a Schauder basis.

$\boxed{2}$ Suppose that $\|.\|_a$ and $\|.\|_b$ are tow norms on a vector space X. We denote by $B_r^a(x)$ and $B_r^b(x)$ the open balls of radius r at $x \in X$ w.r.t. the norms $\|.\|_a$ and $\|.\|_b$, respectively.

a) Show that $\|.\|_a$ and $\|.\|_b$ are equivalent norms if and only if there exists some $r > 0$ such that

$$B_{1/r}^a(0) \subseteq B_1^b(0) \subseteq B_r^a(0).$$

Solution. To say that $\|\cdot\|_a$ and $\|\cdot\|_b$ are equivalent norms means that there exists an $r > 0$ such that $\frac{1}{r}\|\cdot\|_b \leq \|\cdot\|_a \leq r\|\cdot\|_b$[1]. Let us first assume that the norms *are* equivalent, i.e. there exists such a constant r, and prove that $B_{\frac{1}{r}}^a(0) \subset B_1^b(0) \subset B_r^a(0)$[2].

First assume that $\|x\|_b < 1$ for some $x \in X$. Then $\|x\|_a \leq r\|x\|_b \leq r$, which proves that $B_1^b(0) \subset B_r^a(0)$. Then assume that $\|x\|_a < \frac{1}{r}$. If we multiply both sides of the inequality by r and use linearity of the norm, we get $\|rx\|_a < 1$. But by assumption $\frac{1}{r}\|rx\|_b \leq \|rx\|_a < 1$, so $\|x\|_b < 1$. This proves that $B_{\frac{1}{r}}^a(0) \subset B_1^b(0)$.

We now turn to the converse, so assume that $B_{\frac{1}{r}}^a(0) \subset B_1^b(0) \subset B_r^a(0)$ for some $r > 0$. We will show that $\frac{1}{C}\|\cdot\|_a \leq \|\cdot\|_b \leq C\|\cdot\|_a$ for $C = 2r$. Pick $x \in X$. From the linearity of the norm, we find that $\|\frac{x}{2\|x\|_a}\|_a = \frac{1}{2}$, so $\frac{x}{2\|x\|_a} \in B_1^a(0)$, and from our assumption this means that $\|\frac{x}{2\|x\|_a}\|_b < r$. Using the linearity of the norm and multiplying both sides of

[1]The position of a and b in this expression can be switched without altering the content of the definition.

[2]Note that a is in the middle of the norm inequalities, and b is in the middle of the set inclusions.

the inequality with $2\|x\|_a$, we find that $\|x\|_b < 2r\|x\|_a$. The other inequality is proved in a similar way by considering $\|\frac{x}{2r\|x\|_b}\|_b$.

$\boxed{3}$ Let s be the vector space of real-valued sequences $(a_n)_{n\in\mathbb{N}}$. Show that the $\|.\|_1$-norm and $\|.\|_\infty$-norm are not equivalent on s.

Solution. One way of showing that the two norms are not equivalent, is to show that for any positive integer N there is a sequence $a = (a_n)_{n\in\mathbb{N}}$ with $N\|a\|_\infty \leq \|a\|_1$. This would show that the norms are not equivalent, as it would make it impossible to find a constant K such that $\|a\|_1 \leq K\|a\|_\infty$ for any sequence $(a_n)_{n\in\mathbb{N}}$.

Therefore we now fix a positive integer N and look for a sequence $(a_n)_{n\in\mathbb{N}}$ with $N\|a_n\|_\infty \leq \|a_n\|_1$. I claim that the sequence $a = \sum_{n=1}^{N+1} \delta_n$ will do, where δ_n is the standard basis. It consists of 1's in the first $N+1$ places, and then 0's everywhere else. Clearly $\|a\|_1 = N+1$ and $\|a\|_\infty = 1$, and so $N\|a\|_\infty \leq \|a\|_1$. This proves the statement.