Norwegian University of Science and Technology
Department of Mathematical Sciences

Exercise set 7

1 Let $\mathcal{H}$ be a Hilbert space and (x_n) a sequence in $\mathcal{H}$ converging weakly to x. Show that the following statements are equivalent:

1. x_n converges stronlgy to x.
2. $\|x_n\|$ converges to $\|x\|$.

2 Let A be a subset of a Banach space X.

a) Show that A is relatively sequentially compact, then A is bounded.

3 Show the following statement about sets A in a Banach space X.

a) A bounded set A is relatively weakly compact if and only if the weak-* closure of A in X^{**} is in X.

4 Let Y^* be a finite-dimensional subspace of X^*, the dual space of a normed space X. Then there exists a finite set E' in the unit sphere S_X of X such that for every $x^* \in E^*$

$$\max_{x \in E'} |x^*(x)| \geq \|x^*\|/2.$$

Note that $S_X = \{x \in X| \, \|x\| = 1\}$.

5 Give an example of a set Y in a normed space X that is closed but not sequentially weakly closed.