TMA 4230 – PROBLEMS, 2015

Problem 1: Let (X, d_X) and (Y, d_Y) be two metric spaces. Then a function $f : X \to Y$ is called a homeomorphism if it is a continuous bijection with a continuous inverse f^{-1}. In this case one says that (X, d_X) and (Y, d_Y) are homeomorphic.

(a) Suppose a, b are finite real numbers. Show that $(0, 1)$ is homeomorphic to (a, b).

(b) Show that $\mathbb{R}$ is homeomorphic to $(0, 1)$.

(c) Give an example that shows that completeness is not preserved under a homeomorphism.

(d) Show that the notion of a Cauchy sequence is not a topologcial property.

Problem 2: Let X be the set with two points, $\{a, b\}$, with the indiscrete topology. Given an example of a sequence that does not have a unique limit.

Problem 3: Let X be a topological space and S a subset of X. Show that

$$\overline{S} = S \cup \{\text{limit points}\}.$$

Furthermore, show that if a sequence $(x_i)_i$ in S converges in X, then its limit is actually in the closure of S.

Problem 4: Let ℓ^1 be the space of absolutely convergent sequences (x_i), i.e. $\|(x_i)\|_1 = \sum_{i=1}^{\infty} |x_i| < \infty$. We define $e_n = (0, ..., 0, 1, 0, ...)$ to be the sequence which is 1 in the n-th coordinate and 0 elsewhere. Show that the set $\{e_n : n \in \mathbb{N}\}$ is bounded but that it is not totally bounded in ℓ^1.

Problem 5: Let $[a, b]$ be a closed interval in $\mathbb{R}$. Show using the definition of compactness in terms of open covers that $[a, b]$ is compact.

Problem 6: Let X be a topological space and S a subset of X. Show the following statemens:

(a) If X is compact and S is closed in X, then S is compact.

(b) If X is Hausdorff and S is compact, then S is closed in X.

Problem 7: Prove the following assertions:

(a) Let f be a continuous mapping between two topological spaces X and Y. If X is compact, then $f(X)$ is compact in Y.

(b) Let f be continuous mapping between a compact space and $\mathbb{R}$. Then $f(X)$ is contained in some interval $[a, b]$ for finite reals a, b.

(c) Compactness is a topological property.

Problem 8: The **Cantor set** $\mathcal{C}$ is defined as $\mathcal{C} = \cap_{n=1}^{\infty}$, where I_{n+1} is constructed by trisecting I_n and removing the middle third, I_0 being the closed interval $[0, 1]$. Show that $\mathcal{C}$ has the following properties:

(a) $\mathcal{C}$ is non-empty and uncountable.

(b) $\mathcal{C}$ is closed and its complement is dense.

Problem 9: Let X be the normed space $(C[0, 1], \|.\|_1)$, where $C[0, 1]$ is the set of all real-valued continuous functions on $[0, 1]$ and $\|f\|_1 = \int_0^1 |f(t)|dt$. Show that $(C[0, 1], \|.\|_1)$ is not complete.

Problem 10: (One-point compactification)
Let X be a Hausdorff space. Then the *one-point compactification* of X is the space $Y = X \cup \{y_0\}$ for some point y_0 not in X with the following open sets: ($\bullet$) $U \subset X \subset Y$ for any open set $U \subset X$, ($\bullet$) $\{y_0\} \cup (X \backslash K)$ for any compact set K of X.

(a) Show that Y with the above mentioned open sets is a topological space.
(b) Show that Y is compact.
(c) Show that Y is Hausdorff if and only if X is locally compact. Recall: A topological space X is called *locally compact* at a point $x \in X$, if x has a compact neighborhood, and X is called locally compact if it is locally compact at every point.

Problem 11: Let (X, d) be a complete metric space and A a subset of X. Show that (A, d) is complete if and only if A is closed in X with respect to the metric topology.

Problem 12: (Completions of metric spaces) A completion of a metric space (X, d) is a complete metric space $(\widetilde{X}, \widetilde{d})$ and an isometry $i : X \to \widetilde{X}$ such that the image $i(X)$ is dense in $\widetilde{X}$.
Show the following assertions, which will demonstrate that each metric space has a unique completion:

(a) Consider the set of all Cauchy sequences in X. We define two Cauchy sequences $x = (x_k)_k$ and $x = (x_k)_k$ to be equivalent, if $d(x, y)$ converges to 0. Show that this defines an equivalence relation.
(b) Let $\widetilde{X}$ be the set of equivalence classes on the set of Cauchy sequences. Denote by $[x]$ the equivalence class of $[x = (x_k)]$. Define a metric on $\widetilde{X}$ by $\widetilde{d}([x], [y]) := \lim_k d(x_k, y_k)$. Show that $\widetilde{d}$ is well-defined and a metric on $\widetilde{X}$.
(c) Show that there is an isometry $i : X \to \widetilde{X}$ such that $i(X)$ is dense in $\widetilde{X}$.
(d) Suppose a metric space (X, d) has two completions, $(\widetilde{X}, \widetilde{d})$ and $(\widehat{X}, \widehat{d})$, i.e. there exist isometries $i : X \to \widetilde{X}$ and $j : X \to \widehat{X}$. Show that there exists a unique bijective isometry $k : \widehat{X} \to \widetilde{X}$ such that $k \circ i = j$.

Problem 13: Show that the set of bounded sequences ℓ^∞ with the sup-norm $\|x\|_{\ell^\infty} = \sup_k |x_k|$ is a Banach space and that it is not separable.

Problem 14: Let f be a real-valued function on a vector space X. Assume that f satisfies (i) $f(x) \geq 0$ for all $x \in X$, and (ii) $f(\lambda x) = |\lambda| f(x)$ for $\lambda \in \mathbb{R}$. Show:

(a) If f is convex, then the triangle inequality is satisfied and $x \mapsto f(x)$ defines a norm on X.
(b) If the sublevel set $\{x \in X : \ f(x) \leq 1\}$ is convex, then f is a norm on V.

Problem 15: Suppose (X, d) is a complete metric space.

(1) If a set in X is a countable intersection of open sets, then one calls it a G_δ-set. Make sure that you understand that all open sets are G_δ-sets, but not all G_δ-sets are open.
(2) Suppose $(G_n)_{n=1}^\infty$ is a sequences of dense G_δ-sets in X. Then $\cap_{n=1}^\infty G_n$ is also a dense G_δ-set.

Problem 16: Show that for a function $f : \mathbb{R} \to \mathbb{R}$ the points of continuity of f form a G_δ-set. Use the following notion: For any open interval I in $\mathbb{R}$, let the *oscillation* of f

over I be given by

$$\operatorname{osc}_I(f) := \sup\{|f(x) - f(y)| : [x,y] \subset I\}.$$

Try to understand why this definition makes sense and then consider a sequence of positive real numbers $\varepsilon \to 0$ as $n \to \infty$ and define sets

$$U_n = \{x \in \mathbb{R} : \text{There exists an open interval I in } \mathbb{R} \text{ s.t. } x \in I \ \& \ \operatorname{osc}_I(f) < \varepsilon_n\}.$$

Problem 17: Define a function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 0$ for all non-zero x not in $\mathbb{Q}$, $f(0) = 1$ and $f(x) = 1/q$ for all non-zero x in $\mathbb{Q}$ where $x = p/q$ is written in lowest terms.

(a) Show that the previous result on the set of continuity points applied to the function f implies that $\mathbb{R}\backslash\mathbb{Q}$ is a G_δ-set.

(b) Show that there exists no function that has $\mathbb{Q}$ as the set of points of continuity.

Problem 18: (Some variations on Banach-Steinhaus) Let X and Y be Banach spaces and $(T_n)_n$ a sequence in $\mathcal{B}(X,Y)$. Suppose that $\lim_n T_n x$ exists for each $x \in X$. Show that

(a) $\sup_n \|T_n\| < \infty$.

(b) Define the operator $Tx := \lim_n T_n x$ for all $x \in X$. Then T is a bounded linear operator.

Problem 19: Let X and Y be Banach spaces and $(T_n)_n$ a uniformly bounded sequence in $\mathcal{B}(X,Y)$. Show that the set $V = \{x \in X : \ \lim T_n x \text{ exists}\}$ is a closed linear subspace of X.

Problem 20: Let f be a function from $[0,1]$ to $[0,1]$ with closed graph. Show that f is continuous on $[0,1]$.

Problem 21: Let X be a Banach space that contains a dense subspace V which is a G_δ-set. Show that $V = X$.

Problem 22: Let X and Y be Banach spaces. Show that any bounded open map between X and Y may be written as $T = T_0 \circ Q$, where Q is the quotient map from X onto $X/\ker(T)$ and T_0 is an isomorphism between $X/\ker(T)$ and Y.

Problem 23: Find a sequence $c = (c_n)_n$ in ℓ^∞ such that

$$\lim_{n\to\infty} \|c - c^{(n)}\|_\infty = \lim_{n\to\infty} \sup_{k>n} |c_k| \neq 0,$$

where $c^{(n)} = (c_1, ..., c_n, 0, 0, \dots)$.

Problem 24: Let X and Y be normed spaces and T a bounded operator from X to Y. Show that $T^{**}(x) = T(x)$ for all $x \in X$.

Problem 25: Let X be a Banach space and j the natural embedding of X into X''. Show that $j(X)$ is closed.

Problem 26: Let E be a subset of a Hilbert space $\mathcal{H}$. Show that its orthogonal complement $E^\perp$ is a closed linear subspace of $\mathcal{H}$.

Problem 27: Let E be a closed subspace of a normed space X. Show the following: If E and X/E are both complete, then X must be a Banach space.

Problem 28: Let X be a normed linear space. Prove the following: If E is a closed subspace of X and F is a finite-dimensional subspace of X, then $E + F$ is a closed subspace of X.

Problem 29: Let E be a closed subspace of a normed space X. Show the following assertions:
(a) If X is separable, then X/E is separable.
(b) If E and X/E are both separable, then X is separable.

Problem 30: Let $X = \ell^p$ for $p \in [1, \infty]$. Then $E = \{c = (c_k) \in \ell^p : \; c_{2k} = 0 \text{ for all } k \in \mathbb{N}\}$ is a closed subspace of ℓ^p with $\operatorname{codim}(E) = \infty$, $\ell^p \cong E$ and $\ell^p/E \cong \ell^p$.

Problem 31: Let c_0 be the closed subspace of sequences converging to zero in ℓ^∞. Suppose $S = \{(c_k) \in \ell^\infty : \; c_k \in \{0, 1\}\}$. Show the following assertions:
(a) Let c, d be elements of S. Suppose $c_k \neq d_k$ for at most finitely many k. Then $c + c_0 = d + c_0$.
(b) Let c, d be elements of S. Suppose $c_k \neq d_k$ for infinitely many k. Then $\|c - d + c_0\| = 1$.
(c) Use (b) to show that ℓ^∞/c_0 is not separable.

Problem 32: Prove that each Hilbert space has an orthonormal basis by the use of Zorn's lemma.

Problem 33: Let X and Y be normed spaces and T a bounded linear operator from X to Y. If a sequence $\{x_n\}$ in X converges weakly to x in X, show that $Tx_n \to Tx$ weakly.

Problem 34: Let E be a convex subspace of a normed space X. Then show that the weak closure $\overline{E}^w$ is the same as the strong closure $\overline{E}$.

Problem 35: Suppose $\{x_n\}$ converges weakly to x in a normed space X. Show that there is a sequence $\{\tilde{x}_m\}$ of linear combinations of $\{x_n\}$ which converges strongly to x_0. (Hint: Apply the preceding result to $\operatorname{span}(x_n)$).

Problem 36: A weak Cauchy sequence in a normed space X is a sequence $\{x_n\}$ such that $\{x'(x_n)\}$ is a Cauchy sequence for all $x' \in X'$. Show that a weak Cauchy sequence is bounded.

Problem 37: A normed space X is weakly complete if each weak Cauchy sequence in X converges weakly in X. Suppose that X is relfexive. Show that X is weakly complete.

Problem 38: Let c_0 be the space of sequences converging to zero. Show that the continuous linear functionals on c_0 may be identified with ℓ^1.

Problem 39: Let X be a normed space and S a dense subset of X'. Then (x_k) converges weakly to x if and only if $f(x_k) \to f(x)$ for all $f \in S$ and (x_k) is bounded.

Problem 40: Show that weak-convergence of a sequence of functionals (f_k) in X' implies weak-* convergence in X'. (Contemplate over the definitions of these two topologies and which sets of functionals are used to test convergence.)

Problem 41: Let X and Y be normed spaces and T a bounded linear operator between X and Y. Show the following: (i) T is compact if and only if every bounded sequence in X is map onto a sequence with a convergent subsequence. (ii) If T is compact, then every weakly convergent sequence $x_n \to x$ is map onto a strongly convergent one $Tx_n \to Tx$.

Problem 42: Let X be a Banach space and (x_k) a Schauder basis for X. Show that X is separable and that the associated sequence space $E := \{a = (a_k) : \sum_k a_k x_k < \infty\}$ is a Banach space with respect to $\|a\|_E := \sup_n \| \sum_{k=1}^n a_k x_k \|$.

Problem 43: Show that a subset $S \subseteq c_0$ is precompact if and only if there exists a $b \in c_0$ such that for all $a \in S$ one has $|a_k| \leq b_k$ for all $k \in \mathbb{N}$.