## TMA 4230 – PROBLEMS, 2015

**Problem 1:** Let $(X, d_X)$ and $(Y, d_Y)$ be two metric spaces. Then a function $f : X \to Y$ is called a homeomorphism if it is a continuous bijection with a continuous inverse $f^{-1}$. In this case one says that $(X, d_X)$ and $(Y, d_Y)$ are homeomorphic.

(a) Suppose $a, b$ are finite real numbers. Show that $(0, 1)$ is homeomorphic to $(a, b)$.

(b) Show that $\mathbb{R}$ is homeomorphic to $(0, 1)$.

(c) Give an example that shows that completeness is not preserved under a homeomorphism.

(d) Show that the notion of a Cauchy sequence is not a topologcial property.

**Problem 2:** Let $X$ be the set with two points, $\{a, b\}$, with the indiscrete topology. Given an example of a sequence that does not have a unique limit.

**Problem 3:** Let $X$ be a topological space and $S$ a subset of $X$. Show that

$$\overline{S} = S \cup \{\text{limit points}\}.$$

Furthermore, show that if a sequence $(x_i)_i$ in $S$ converges in $X$, then its limit is actually in the closure of $S$.

**Problem 4:** Let $\ell^1$ be the space of absolutely convergent sequences $(x_i)$, i.e. $\|(x_i)\|_1 = \sum_{i=1}^{\infty} |x_i| < \infty$. We define $e_n = (0, ..., 0, 1, 0, ...)$ to be the sequence which is 1 in the n-th coordinate and 0 elsewhere. Show that the set $\{e_n : n \in \mathbb{N}\}$ is bounded but that it is not totally bounded in $\ell^1$.

**Problem 5:** Let $[a, b]$ be a closed interval in $\mathbb{R}$. Show using the definition of compactness in terms of open covers that $[a, b]$ is compact.

**Problem 6:** Let $X$ be a topological space and $S$ a subset of $X$. Show the following statemens:

(a) If $X$ is compact and $S$ is closed in $X$, then $S$ is compact.

(b) If $X$ is Hausdorff and $S$ is compact, then $S$ is closed in $X$.

**Problem 7:** Prove the following assertions:

(a) Let $f$ be a continuous mapping between two topological spaces $X$ and $Y$. If $X$ is compact, then $f(X)$ is compact in $Y$.

(b) Let $f$ be continuous mapping between a compact space and $\mathbb{R}$. Then $f(X)$ is contained in some interval $[a, b]$ for finite reals $a, b$.

(c) Compactness is a topological property.

**Problem 8:** The **Cantor set** $\mathcal{C}$ is defined as $\mathcal{C} = \cap_{n=1}^{\infty}$, where $I_{n+1}$ is constructed by trisecting $I_n$ and removing the middle third, $I_0$ being the closed interval $[0, 1]$. Show that $\mathcal{C}$ has the following properties:

(a) $\mathcal{C}$ is non-empty and uncountable.

(b) $\mathcal{C}$ is closed and its complement is dense.

**Problem 9:** Let $X$ be the normed space $(C[0, 1], \|.\|_1)$, where $C[0, 1]$ is the set of all real-valued continuous functions on $[0, 1]$ and $\|f\|_1 = \int_0^1 |f(t)|dt$. Show that $(C[0, 1], \|.\|_1)$ is not complete.

**Problem 10:** (One-point compactification)
Let $X$ be a Hausdorff space. Then the *one-point compactification* of $X$ is the space $Y = X \cup \{y_0\}$ for some point $y_0$ not in $X$ with the following open sets: ($\bullet$) $U \subset X \subset Y$ for any open set $U \subset X$, ($\bullet$) $\{y_0\} \cup (X \backslash K)$ for any compact set $K$ of $X$.

(a) Show that $Y$ with the above mentioned open sets is a topological space.
(b) Show that $Y$ is compact.
(c) Show that $Y$ is Hausdorff if and only if $X$ is locally compact. Recall: A topological space $X$ is called *locally compact* at a point $x \in X$, if $x$ has a compact neighborhood, and $X$ is called locally compact if it is locally compact at every point.

**Problem 11:** Let $(X, d)$ be a complete metric space and $A$ a subset of $X$. Show that $(A, d)$ is complete if and only if $A$ is closed in $X$ with respect to the metric topology.

**Problem 12:** (Completions of metric spaces) A completion of a metric space $(X, d)$ is a complete metric space $(\widetilde{X}, \widetilde{d})$ and an isometry $i : X \to \widetilde{X}$ such that the image $i(X)$ is dense in $\widetilde{X}$.
Show the following assertions, which will demonstrate that each metric space has a unique completion:

(a) Consider the set of all Cauchy sequences in $X$. We define two Cauchy sequences $x = (x_k)_k$ and $x = (x_k)_k$ to be equivalent, if $d(x, y)$ converges to 0. Show that this defines an equivalence relation.
(b) Let $\widetilde{X}$ be the set of equivalence classes on the set of Cauchy sequences. Denote by $[x]$ the equivalence class of $[x = (x_k)]$. Define a metric on $\widetilde{X}$ by $\widetilde{d}([x], [y]) := \lim_k d(x_k, y_k)$. Show that $\widetilde{d}$ is well-defined and a metric on $\widetilde{X}$.
(c) Show that there is an isometry $i : X \to \widetilde{X}$ such that $i(X)$ is dense in $\widetilde{X}$.
(d) Suppose a metric space $(X, d)$ has two completions, $(\widetilde{X}, \widetilde{d})$ and $(\widehat{X}, \widehat{d})$, i.e. there exist isometries $i : X \to \widetilde{X}$ and $j : X \to \widehat{X}$. Show that there exists a unique bijective isometry $k : \widehat{X} \to \widetilde{X}$ such that $k \circ i = j$.

**Problem 13:** Show that the set of bounded sequences $\ell^\infty$ with the sup-norm $\|x\|_{\ell^\infty} = \sup_k |x_k|$ is a Banach space and that it is not separable.

**Problem 14:** Let $f$ be a real-valued function on a vector space $X$. Assume that $f$ satisfies (i) $f(x) \geq 0$ for all $x \in X$, and (ii) $f(\lambda x) = |\lambda| f(x)$ for $\lambda \in \mathbb{R}$. Show:

(a) If $f$ is convex, then the triangle inequality is satisfied and $x \mapsto f(x)$ defines a norm on $X$.
(b) If the sublevel set $\{x \in X : \ f(x) \leq 1\}$ is convex, then $f$ is a norm on $V$.

**Problem 15:** Suppose $(X, d)$ is a complete metric space.

(1) If a set in $X$ is a countable intersection of open sets, then one calls it a $G_\delta$-set. Make sure that you understand that all open sets are $G_\delta$-sets, but not all $G_\delta$-sets are open.
(2) Suppose $(G_n)_{n=1}^\infty$ is a sequences of dense $G_\delta$-sets in $X$. Then $\cap_{n=1}^\infty G_n$ is also a dense $G_\delta$-set.

**Problem 16:** Show that for a function $f : \mathbb{R} \to \mathbb{R}$ the points of continuity of $f$ form a $G_\delta$-set. Use the following notion: For any open interval $I$ in $\mathbb{R}$, let the *oscillation* of $f$

over $I$ be given by

$$\operatorname{osc}_I(f) := \sup\{|f(x) - f(y)| : [x, y] \subset I\}.$$

Try to understand why this definition makes sense and then consider a sequence of positive real numbers $\varepsilon \to 0$ as $n \to \infty$ and define sets

$$U_n = \{x \in \mathbb{R} : \text{There exists an open interval I in } \mathbb{R} \text{ s.t. } x \in I \ \& \ \operatorname{osc}_I(f) < \varepsilon_n\}.$$

**Problem 17:** Define a function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 0$ for all non-zero $x$ not in $\mathbb{Q}$, $f(0) = 1$ and $f(x) = 1/q$ for all non-zero $x$ in $\mathbb{Q}$ where $x = p/q$ is written in lowest terms.

(a) Show that the previous result on the set of continuity points applied to the function $f$ implies that $\mathbb{R} \backslash \mathbb{Q}$ is a $G_\delta$-set.

(b) Show that there exists no function that has $\mathbb{Q}$ as the set of points of continuity.

**Problem 18:** (Some variations on Banach-Steinhaus) Let $X$ and $Y$ be Banach spaces and $(T_n)_n$ a sequence in $\mathcal{B}(X, Y)$. Suppose that $\lim_n T_n x$ exists for each $x \in X$. Show that

(a) $\sup_n \|T_n\| < \infty$.

(b) Define the operator $Tx := \lim_n T_n x$ for all $x \in X$. Then $T$ is a bounded linear operator.

**Problem 19:** Let $X$ and $Y$ be Banach spaces and $(T_n)_n$ a sequence in $\mathcal{B}(X, Y)$. Show that the set $V = \{x \in X : \lim T_n x \text{ exists}\}$ is a closed linear subspace of $X$.

**Problem 20:** Let $f$ be a function from $[0, 1]$ to $[0, 1]$ with closed graph. Show that $f$ is continuous on $[0, 1]$.

**Problem 21:** Let $X$ be a Banach space that contains a dense subspace $V$ which is a $G_\delta$-set. Show that $V = X$.

**Problem 22:** Let $X$ and $Y$ be Banach spaces. Show that any bounded open map between $X$ and $Y$ may be written as $T = T_0 \circ Q$, where $Q$ is the quotient map from $X$ onto $X/\ker(T)$ and $T_0$ is an isomorphism between $X/\ker(T)$ and $Y$.

**Problem 23:** Find a sequence $a = (a_n)_n$ in $\ell^\infty$ such that

$$\lim_{n\to\infty} \|c - c^{(n)}\|_\infty = \lim_{n\to\infty} \sup_{k>n} |c_k| \neq 0,$$

where $c^{(n)} = (c_1, ..., c_n, 0, 0, \dots)$.

**Problem 24:** Let $X$ and $Y$ be normed spaces and $T$ a bounded operator from $X$ to $Y$. Show that $T^{**}(x) = T(x)$ for all $x \in X$.