TMA 4230 - PROBLEMS, 2015

Problem 1: Let (X, d_X) and (Y, d_Y) be two metric spaces. Then a function $f : X \to Y$ is called a homeomorphism if it is a continuous bijection with a continuous inverse f^{-1} . In this case one says that (X, d_X) and (Y, d_Y) are homeomorphic.

- (a) Suppose a, b are finite real numbers. Show that (0, 1) is homeomorphic to (a, b).
- (b) Show that \mathbb{R} is homeomorphic to (0, 1).
- (c) Give an example that shows that completeness is not preserved under a homeomorphism.
- (d) Show that the notion of a Cauchy sequence is not a topological property.
- **Problem 2:** Let X be the set with two points, $\{a, b\}$, with the indiscrete topology. Given an example of a sequence that does not have a unique limit.
- **Problem 3:** Let X be a topological space and S a subset of X. Show that

 $\overline{S} = S \cup \{\text{limit points}\}.$

Furthermore, show that if a sequence $(x_i)_i$ in S converges in X, then its limit is actually in the closure of S.

- **Problem 4:** Let ℓ^1 be the space of absolutely convergent sequences (x_i) , i.e. $||(x_i)||_1 = \sum_{i=1}^{\infty} |x_i| < \infty$. We define $e_n = (0, ..., 0, 1, 0, ...)$ to be the sequence which is 1 in the n-th coordinate and 0 elsewhere. Show that the set $\{e_n : n \in \mathbb{N}\}$ is bounded but that it is not totally bounded in ℓ^1 .
- **Problem 5:** Let [a, b] be a closed interval in \mathbb{R} . Show using the definition of compactness in terms of open covers that [a, b] is compact.
- **Problem 6:** Let X be a topological space and S a subset of X. Show the following statemens: (a) If X is compact and S is closed in X, then S is compact.
 - (b) If X is Hausdorff and S is compact, then S is closed in X.
- **Problem 7:** Prove the following assertions:
 - (a) Let f be a continuous mapping between two topological spaces X and Y. If X is compact, then f(X) is compact in Y.
 - (b) Let f be continuous mapping between a compact space and \mathbb{R} . Then f(X) is contained in some interval [a, b] for finite reals a, b.
 - (c) Compactness is a topological property.
- **Problem 8:** The **Cantor set** C is defined as $C = \bigcap_{n=1}^{\infty}$, where I_{n+1} is constructed by trisecting I_n and removing the middle third, I_0 being the closed interval [0, 1]. Show that C has the following properties:
 - (a) C is non-empty and uncountable.
 - (b) C is closed and its complement is dense.
- **Problem 9:** Let X be the normed space $(C[0,1], \|.\|_1)$, where C[0,1] is the set of all realvalued continuous functions on [0,1] and $\|f\|_1 = \int_0^1 |f(t)| dt$. Show that $(C[0,1], \|.\|_1)$ is not complete.

Problem 10: (One-point compactification)

Let X be a Hausdorff space. Then the one-point compactification of X is the space $Y = X \cup \{y_0\}$ for some point y_0 not in X with the following open sets: (•) $U \subset X \subset Y$ for any open set $U \subset X$, (•) $\{y_0\} \cup (X \setminus K)$ for any compact set K of X.

- (a) Show that Y with the above mentioned open sets is a topological space.
- (b) Show that Y is compact.
- (c) Show that Y is Hausdorff if and only if X is locally compact. Recall: A topological space X is called *locally compact* at a point $x \in X$, if x has a compact neighborhood, and X is called locally compact if it is locally compact at every point.
- **Problem 11:** Let (X, d) be a complete metric space and A a subset of X. Show that (A, d) is complete if and only if A is closed in X with respect to the metric topology.
- **Problem 12:** (Completions of metric spaces) A completion of a metric space (X, d) is a complete metric space $(\widetilde{X}, \widetilde{d})$ and an isometry $i : X \to \widetilde{X}$ such that the image i(X) is dense in \widetilde{X} .

Show the following assertions, which will demonstrate that each metric space has a unique completion:

- (a) Consider the set of all Cauchy sequences in X. We define two Cauchy sequences $x = (x_k)_k$ and $x = (x_k)_k$ to be equivalent, if d(x, y) converges to 0. Show that this defines an equivalence relation.
- (b) Let X be the set of equivalence classes on the set of Cauchy sequences. Denote by [x] the equivalence class of $[x = (x_k)]$. Define a metric on \widetilde{X} by $\widetilde{d}([x], [y]) := \lim_k d(x_k, y_k)$. Show that \widetilde{d} is well-defined and a metric on \widetilde{X} .
- (c) Show that there is an isometry $i: X \to \widetilde{X}$ such that i(X) is dense in \widetilde{X} .
- (d) Suppose a metric space (X, d) has two completions, $(\widetilde{X}, \widetilde{d})$ and $(\widehat{X}, \widehat{d})$, i.e. there exist isometries $i: X \to \widetilde{X}$ and $j: X \to \widehat{X}$. Show that there exists a unique bijective isometry $k: \widehat{X} \to \widetilde{X}$ such that $k \circ i = j$.
- **Problem 13:** Show that the set of bounded sequences ℓ^{∞} with the sup-norm $||x||_{\ell^{\infty}} = \sup_{k} |x_{k}|$ is a Banach space and that it is not separable.
- **Problem 14:** Let f be a real-valued function on a vector space X. Assume that f satisfies (i) $f(x) \ge 0$ for all $x \in X$, and (ii) $f(\lambda x) = |\lambda| f(x)$ for $\lambda \in \mathbb{R}$. Show:
 - (a) If f is convex, then the triangle inequality is satisfied and $x \mapsto f(x)$ defines a norm on X.
 - (b) If the sublevel set $\{x \in X : f(x) \le 1\}$ is convex, then f is a norm on V.

Problem 15: Suppose (X, d) is a complete metric space.

- (1) If a set in X is a countable intersection of open sets, then one calls it a G_{δ} -set. Make sure that you understand that all open sets are G_{δ} -sets, but not all G_{δ} -sets are open.
- (2) Suppose $(G_n)_{n=1}^{\infty}$ is a sequences of dense G_{δ} -sets in X. Then $\bigcap_{n=1}^{\infty} G_n$ is also a dense G_{δ} -set.
- **Problem 16:** Show that for a function $f : \mathbb{R} \to \mathbb{R}$ the points of continuity of f form a G_{δ} -set. Use the following notion: For any open interval I in \mathbb{R} , let the *oscillation* of f

over I be given by

$$\operatorname{osc}_{I}(f) := \sup\{|f(x) - f(y)| : [x, y] \subset I\}.$$

Try to understand why this definition makes sense and then consider a sequence of positive real numbers $\varepsilon \to 0$ as $n \to \infty$ and define sets

 $U_n = \{x \in \mathbb{R} : \text{There exists an open interval I in } \mathbb{R} \text{ s.t. } x \in I \& \operatorname{osc}_I(f) < \varepsilon_n \}.$

Problem 17: Define a function $f : \mathbb{R} \to \mathbb{R}$ by f(x) = 0 for all non-zero x not in \mathbb{Q} , f(0) = 1

- and f(x) = 1/q for all non-zero x in \mathbb{Q} where x = p/q is written in lowest terms.
- (a) Show that the previous result on the set of continuity points applied to the function f implies that $\mathbb{R}\setminus\mathbb{Q}$ is a G_{δ} -set.
- (b) Show that there exists no function that has \mathbb{Q} as the set of points of continuity.
- **Problem 18:** (Some variations on Banach-Steinhaus) Let X and Y be Banach spaces and $(T_n)_n$ a sequence in $\mathcal{B}(X,Y)$. Suppose that $\lim_n T_n x$ exists for each $x \in X$. Show that
 - (a) $\sup_n ||T_n|| < \infty$.
 - (b) Define the operator $Tx := \lim_{n \to \infty} T_n x$ for all $x \in X$. Then T is a bounded linear operator.
- **Problem 19:** Let X and Y be Banach spaces and $(T_n)_n$ a sequence in $\mathcal{B}(X, Y)$. Show that the set $V = \{x \in X : \lim T_n x \text{ exists}\}$ is a closed linear subspace of X.
- **Problem 20:** Let f be a function from [0,1] to [0,1] with closed graph. Show that f is continuous on [0,1].
- **Problem 21:** Let X be a Banach space that contains a dense subspace V which is a G_{δ} -set. Show that V = X.
- **Problem 22:** Let X and Y be Banach spaces. Show that any bounded open map between X and Y may be written as $T = T_0 \circ Q$, where Q is the quotient map from X onto $X/\ker(T)$ and T_0 is an isomorphism between $/\ker(T)$ and Y.
- **Problem 23:** Find a sequence $a = (a_n)_n$ in ℓ^{∞} such that

$$\lim_{n \to \infty} \|c - c^{(n)}\|_{\infty} = \lim_{n \to \infty} \sup_{k > n} |c_k| \neq 0,$$

where $c^{(n)} = (c_1, ..., c_n, 0, 0, ...).$

Problem 24: Let X and Y be normed spaces and T a bounded operator from X to Y. Show that $T^{**}(x) = T(x)$ for all $x \in X$.