Lecture Notes in

TMA4145 - Linear Methods

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`https://wiki.math.ntnu.no/users/ehrnstro/teaching/linearmethods`

Contents

Chapter 1

Sets, functions and real numbers

1.1 Sets

Basic definitions

A **set** is a collection of elements, such as

$$\{1, 2, 3\}, \quad \{a, b, \dagger, \ddagger\}, \quad \text{or} \quad \{\text{all yellow horses}\}.$$

Sets are unordered. Two sets are **equal** if they contain the same elements,

$$\{1, 2, 3\} = \{3, 2, 1\},$$

whence the set containing no elements,

$$\emptyset = \{\}$$

is unique; it is called the **empty set**.

The **cardinality** of a finite set is its number of elements:

$$|\{a, b\}| = 2 \quad \text{and} \quad |\emptyset| = 0.$$

Ex. Some well-known infinite sets are the **natural numbers**,[1]

$$\mathbb{N} = \{1, 2, 3, \ldots\},$$

the **integers**,

$$\mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\},$$

and the **real**, $\mathbb{R}$, and **complex numbers**, $\mathbb{C}$.

1.2 Membership and inclusions

Membership (possessive relations)

If x is an element in a set A we write

$$x \in A \quad \text{or} \quad A \ni x,$$

[1] In some textbooks also the zero element is included in the set of natural numbers.

and if not

$$x \notin A \quad \text{or} \quad A \not\ni x.$$

Ex.

- The **rationals** can be constructed from elements in $\mathbb{Z}$ and $\mathbb{N}$:

$$\mathbb{Q} = \{a/b \colon a \in \mathbb{Z}, b \in \mathbb{N}\}$$

- $\sqrt{2}$ is a real number, but not a rational one:

$$\sqrt{2} \in \mathbb{R}, \qquad \sqrt{2} \notin \mathbb{Q}.$$

Quantifiers

Quantifiers are used to abbreviate notation. The most important ones are:

- $\forall$ Universal quantifier: 'For any','for all'
- $\exists$ Existential quantifier: 'There exists'
- ! Uniqueness quantifier: 'a unique'

Ex. For any real number x there exists a unique real number $-x$ with the property that the sum of x and $-x$ is zero:

$$\forall\, x \in \mathbb{R} \qquad \exists!\ (-x) \in \mathbb{R}; \qquad x + (-x) = 0.$$

Inclusions

A set A is a **subset** of a set B if any element in A is also an element in B:

$$A \subset B \quad (\text{or } A \subseteq B) \quad \overset{\text{def.}}{\iff} \quad [x \in A \Rightarrow x \in B]$$

A subset $A \subset B$ can also be a **proper subset** of B:

$$A \subsetneq B \quad \overset{\text{def.}}{\iff} \quad A \subset B \text{ but } A \neq B.$$

Ex.

- The natural numbers is a subset of the set of non-negative integers, which is a proper subset of the set of real numbers:

$$\mathbb{N} \subset \{0, 1, 2, \ldots\} \subsetneq \mathbb{R}.$$

- The empty set is a subset of any other set (including itself):

$$\emptyset \subset \emptyset \subset \{1, 2, 3\}.$$

- The continuously differentiable real-valued functions on the real line is a subset of the continuous functions:

$$C^1(\mathbb{R}, \mathbb{R}) \subset C(\mathbb{R}, \mathbb{R}).$$

1.3 Set operations

Unions and intersections

The **union** of two sets A and B is the set of elements that are either in A or in B:

$$A \cup B \overset{\text{def.}}{=} \{x \colon x \in A \text{ or } x \in B\}.$$

Their **intersection** is the collection of elements belonging to both A and B:

$$A \cap B \stackrel{\text{def.}}{=} \{x \colon x \in A \text{ and } x \in B\}.$$

Ex.

- For finite sets:

$$\{A, B, C\} \cup \{A, C, D\} = \{A, B, C, D\}, \qquad \{A, B, C\} \cap \{A, C, D\} = \{A, C\}.$$

- For two intervals:

$$(-\infty, 1) \cup (0, \infty) = \mathbb{R}, \qquad (-\infty, 1) \cap (0, \infty) = (0, 1).$$

- For any set A,

$$A \cup \emptyset = A, \qquad A \cap \emptyset = \emptyset.$$

Set differences and complements

The **relative complement** (or **set difference**) of A in B contains any element in B not in A:

$$B \setminus A \stackrel{\text{def.}}{=} \{x \colon x \in B \text{ and } x \notin A\}.$$

When B is understood to be known, this can also be expressed as $\complement(A)$ or $\operatorname{comp}(A)$, the **complement** of A (in B).

Ex. The complement of the unit ball in three-dimensional Euclidean space is the set of vectors of unit length or larger:

$$\operatorname{comp}(\{x \in \mathbb{R}^3 \colon |x| < 1\}) = \mathbb{R}^3 \setminus \{x \in \mathbb{R}^3 \colon |x| < 1\} = \{x \in \mathbb{R}^3 \colon |x| \geq 1\}.$$

1.4 Relations

Cartesian products

The **Cartesian product** of two sets A and B is the set of **ordered pairs** (a, b) of elements $a \in A$ and $b \in B$:

$$A \times B \stackrel{\text{def.}}{=} \{(a, b) \colon a \in A, b \in B\}.$$

Ex.

- The Cartesian product of $\{1, 2\}$ and $\{\dagger, \ddagger\}$ has four elements:[1]

$$\{1, 2\} \times \{\dagger, \ddagger\} = \{(1, \dagger), (1, \ddagger), (2, \dagger), (2, \ddagger)\}.$$

- The Cartesian product of the set of points on the real line and the set of points in the plane is the set of points in three-dimensional space:

$$\mathbb{R} \times \mathbb{R}^2 = \mathbb{R}^3.$$

Relations

A **relation** (or **binary relation**) on two sets A and B is a subset G of $A \times B$:

$$G = \{(a, b) \in A \times B \colon a \text{ satisfying some criteria}, b \text{ satisfying some criteria}\}$$

[1] In general, $|A \times B| = |A||B|$ for finite sets.

The set A is called the relation's **domain**, B its **codomain**, and G its **graph**. The graph of a relation can most easily be thought of as connections (**edges**) between 'points' in A and B (**vertices**).

Ex.

- Students enlisted for courses at a university is a relation on the set of students and the set of courses. For example,

$$\{(\text{Anna, Math}), (\text{Anna, Chem}), (\text{Niels, Math}), (\text{Niels, Lit})\}$$

  is the graph of a relation on the domain $\{\text{Anna, Niels}\}$ and the codomain $\{\text{Math, Chem, Lit}\}$.
- Relations can be defined on a product set $A \times A$. For example, '$\leq$' is a relation on $\mathbb{R} \times \mathbb{R}$, whose graph is determined by

$$(a, b) \in G \quad \Longleftrightarrow \quad a \leq b.$$

Functions

A **function** (or **mapping**) is a relation with the property that for every a in its domain there is a unique b in its codomain such that (a, b) is in the graph.

$$\forall\, a \in A \quad \exists!\, b \in B; \quad (a, b) \in G.$$

To indicate this, one often writes x, y and X, Y instead of a, b and A, B. Although functions are completely described by their graphs, it is common to use an extra letter, such as f, to express functional relations. One writes

$$f\colon X \to Y, \qquad x \mapsto f(x)$$

or simply

$$y = f(x)$$

to indicate the **argument**, x, and **value**, y, of a function.[1]

Ex.

- The relation with graph

$$G = \{(x, y) \in \mathbb{R} \times \mathbb{R}\colon y = x^2\}$$

  defines a function $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$.
- The length of a two-vector

$$|\cdot|\colon \mathbb{R}^2 \to [0, \infty), \qquad (x_1, x_2) \mapsto (x_1^2 + x_2^2)^{1/2}$$

  is a function from the set of vectors in the plane to the set of non-negative real numbers.

Sequences

A function with the domain $\mathbb{N}$ is called a **sequence**, instead of writing $f : \mathbb{N} \to X$ one writes $\{x_n\}_{n\in\mathbb{N}}$ (or simply $\{x_n\}$) to denote the sequence $n \longmapsto x_n$.

Ex.

- $\{1/n\}$ is a sequence in $\mathbb{Q}$
- $\{f_n(t) = t^n\}$ is a sequence in $BC([0, 1], \mathbb{R})$

N.b. There is a difference between a sequence $\{x_n\}$ of points in some set M and a subset

[1] Note the difference between the *function*, written f, $f(\cdot)$, or $x \mapsto f(x)$, and its *value*, $f(x)$, at a particular point x.

$\{x_n\}_{n\in\mathbb{N}} \subset M$. In a sequence points may coincide $x_n = x_m$ when $n \neq m$, in a set $\{x_n\}_{n\in\mathbb{N}}$ we assume that $x_n \neq x_m$ when $n \neq m$.

1.5 Invertibility

Range and surjectivity

The **range** (or **image**) of a function $f\colon X \to Y$ is the set of elements $y \in Y$ in its codomain for which there is an $x \in X$ in its domain such that $y = f(x)$,

$$\operatorname{ran}(f) \overset{\text{def.}}{=} \{f(x)\colon x \in X\}.$$

A function is **surjective** (or **onto**) if its range equals its codomain,

$$f \text{ surjective} \quad \overset{\text{def.}}{\iff} \quad \operatorname{ran}(f) = Y.$$

This is the same as that, for every $y \in Y$, there is an $x \in X$ with $f(x) = y$.

Ex.

- The range of the function
$$f\colon \mathbb{R} \to \mathbb{R}, \qquad x \mapsto x^2$$
is $\operatorname{ran}(f) = [0, \infty)$, whence it is *not* surjective.
- Defined differently,
$$f\colon \mathbb{R} \to [0, \infty), \qquad x \mapsto x^2$$
is surjective.
- The differential operator $\frac{d}{dx}\colon C^1(\mathbb{R}, \mathbb{R}) \to C(\mathbb{R}, \mathbb{R})$ is surjective, since for any $f \in C(\mathbb{R}, \mathbb{R})$ there exists $F = \left[x \mapsto \int_0^x f(t)\,dt\right] \in C^1(\mathbb{R}, \mathbb{R})$ such that $\frac{d}{dx}F = f$.

Injectivity

A function is **injective** (or **one-to-one**) if different elements in its domain are mapped onto different elements in its codomain,

$$f \text{ injective} \quad \overset{\text{def.}}{\iff} \quad [f(x_1) = f(x_2) \implies x_1 = x_2]$$

Put differently, for any $y \in Y$ there is at most one $x \in X$ with $f(x) = y$.

Ex.

- The function
$$f\colon \mathbb{R} \to \mathbb{R}, \qquad x \mapsto x^2$$
is not injective, since $x^2 = (-x)^2$.
- The function
$$f\colon \mathbb{N} \to \mathbb{N}, \qquad n \mapsto 2n$$
that assigns to each natural number twice its value is injective, since
$$2m = 2n \quad \implies \quad m = n, \qquad m, n \in \mathbb{N}.$$

- The differential operator $\frac{d}{dx}\colon C^1(\mathbb{R},\mathbb{R}) \to C(\mathbb{R},\mathbb{R})$ is not injective, since, for $f \in C^1(\mathbb{R},\mathbb{R})$,

$$\frac{d}{dx}(f(x)+c) = \frac{d}{dx}f(x) \quad \text{for all} \quad c \in \mathbb{R}.$$

Invertibility

A function that is both injective and surjective is called **bijective** (or **invertible**). Since its graph covers the entire codomain (surjectivity), and since for each $y \in Y$ there is exactly one $x \in X$ with $f(x) = y$ (injectivity), there exists a function

$$f^{-1}\colon Y \to X, \qquad y \mapsto x,$$

called the **inverse of** f. An invertible function satisfies

$$f^{-1}(f(x)) = x \quad \text{for all} \quad x \in X \qquad \text{and} \qquad f(f^{-1}(y)) = y \quad \text{for all} \quad y \in Y,$$

or, shorter,

$$f^{-1} \circ f = \mathrm{id}_X \quad \text{and} \quad f \circ f^{-1} = \mathrm{id}_Y.$$

N.b. An injection is always invertible on its range, but not necessarily on the entire codomain.

Ex.

- The function $x \mapsto x^2$ is invertible on $[0,\infty)$ (which is easily seen from its graph).
- The map $n \mapsto 2n$, $\mathbb{N} \to 2\mathbb{N}$, is a bijection between the set of positive natural numbers and the set of even numbers. In this sense the cardinality of the set of natural numbers and the cardinality of the set of even numbers are the same.
- The function defined by

  $$f(a,b) = a + b\mathrm{i}$$

  is a bijection $\mathbb{R}^2 \to \mathbb{C}$, from the real onto the complex plane.
- One can prove that there is no invertible function from $\mathbb{N}$ to $\mathbb{R}$. In this sense, the cardinality of the real numbers is greater than that of the natural numbers (the natural numbers are said to be **countable**, whereas the real numbers are **uncountable**).
- The differential operator

  $$1 - \partial_x^2\colon C^\infty_{2\pi\text{-per}}(\mathbb{R},\mathbb{R}) \to C^\infty_{2\pi\text{-per}}(\mathbb{R},\mathbb{R})$$

  from the set of 2π-**periodic infinitely differentiable real-valued functions** onto itself is a bijection. The operator $1 + \partial_x^2$ on the same set of functions is *not* (can you see why?). This means that the differential equation

  $$f'' - f = 0$$

  has exactly one 2π-periodic solution (namely $f \equiv 0$), whereas the equation

  $$f'' + f = 0$$

  has many.
- A major question in linear algebra is: When is the matrix A in an equation

  $$Ax = b$$

  invertible? Here, A is seen as an operator $\mathbb{R}^n \to \mathbb{R}^n$, mapping vectors onto vectors.

1.6 Real numbers

Operations

The usual arithmetic operations on the set of real numbers, addition and multiplication, are functions from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$. The real numbers with these operations satisfy the following rules ($(\mathbb{R}, +, \cdot)$ is a field). There exist two special elements called zero (0) and one (1) such that for any real numbers a, b, c

name	addition	multiplication
associativity	$(a+b)+c = a+(b+c)$	$(a \cdot b) \cdot c = a \cdot (b \cdot c)$
commutativity	$a+b = b+a$	$a \cdot b = b \cdot a$
identity elements	$a+0 = a$	$a \cdot 1 = a$
inverses	$\exists(-a) \in \mathbb{R};\ a+(-a) = 0$	for $a \neq 0, \exists a^{-1} \in \mathbb{R};\ a \cdot a^{-1} = 1$
distributivity	$(a+b) \cdot c = a \cdot c + b \cdot c$	

Order

Another important notion is order. It allows to compare real numbers and say that one is larger than the other. Formally, the total order on the set of real numbers is given by a relation $G \subset \mathbb{R} \times \mathbb{R}$ that enjoys the following properties ($(a, b) \in G$ if and only if $a \geq b$)

- Antisymmetry: $(a, b) \in G$ and $(b, a) \in G$ implies $a = b$
- Transitivity: $(a, b) \in G$ and $(b, c) \in G$ implies $(a, c) \in G$
- Totality: For any $a \in G$ and $b \in G$ either $(a, b) \in G$ or $(b, a) \in G$

The usual order on $\mathbb{R}$ is connected to the arithmetic operations in the following way

- $(a, b) \in G \implies (a + c, b + c) \in G$ for $\forall c \in \mathbb{R}$
- $(a, 0) \in G$ and $(b, 0) \in G \implies (a + b, 0) \in G$ and $(a \cdot b, 0) \in G$

Ex.

- The order axioms above imply $(a, a) \in G$ for any $a \in \mathbb{R}$.
- By the multiplicative property of 1 and the relations of the order and arithmetic operations, one can see that $(1, 0) \in G$.

Supremum

An **upper bound** of a set $S \subset \mathbb{R}$ is a real number b such that $\forall a \in S, a \leq b$. A real number s is called a **least upper bound** or **supremum** for S if s is an upper bound and $s \leq b$ for any upper bound b for S.

The **completeness axiom** for real numbers says that if a non-empty set $S \subset \mathbb{R}$ has an upper bound then it has a least upper bound. The uniqueness of the lest upper bound follows from the properties of order and we denote this least upper bound by $\sup S$. Note that the set of rational numbers $\mathbb{Q}$ satisfies all the properties discussed above but the completeness axiom.

If a set $S \subset \mathbb{R}$ has no upper bound then $\sup S = +\infty$, one furthermore defines $\sup(\emptyset) = -\infty$. Thus any subset of $\mathbb{R}$ has a supremum in $\{-\infty\} \cup \mathbb{R} \cup +\infty$. In a similar fashion, the **infimum** is defined as the greatest lower bound; it can be also defined by $\inf(S) = -\sup\{-a \in \mathbb{R} : a \in S\}$.

Ex.

- Let $S = \{x \in \mathbb{R} : x^2 \leq 2\}$. Then $\sup(S) = \sqrt{2}$ and $\inf(S) = -\sqrt{2}$.
- If we consider $S_0 = \mathbb{Q} \cup S$ then $\sup(S_0) = \sup(S) \notin \mathbb{Q}$. This is the fundamental difference between the sets of rational and real numbers.

An accurate construction of real numbers (from rational ones) was given by Richard Dedekind in 1870s.

Chapter 2

Metric spaces

2.1 Open and closed sets

Definition
Let X be a set and $d\colon X \times X \to [0, \infty)$ a function such that

(i)	$d(x, y) = d(y, x)$,	(symmetry)
(ii)	$d(x, y) \leq d(x, z) + d(z, y)$,	(triangle inequality)
(iii)	$d(x, y) = 0 \quad$ if only if $\quad x = y$.	(non-degeneracy)

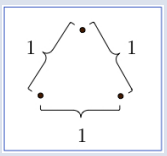
Then the pair (X, d) is called a **metric space** and the function d is called a **metric** or **distance** on X.

A subset $M \subset X$ is called a **subspace** of X, written $(M, d) \subset (X, d)$, if M is endowed with the same metric as X, called the **induced metric** on M. Subspaces of metric spaces are themselves metric spaces.

Ex.

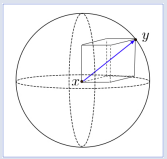
- Any subset S of $\mathbb{R}$ endowed with the usual distance $d(x, y) = |x - y|$.
- Any set becomes a metric space when endowed with the **discrete metric**

$$d(x, y) := \begin{cases} 1, & x \neq y, \\ 0, & x = y. \end{cases}$$



- $\mathbb{R}^n$ becomes a metric space when endowed with the **Euclidean distance**

$$d(x, y) := |x - y| = \left((x_1 - y_1)^2 + \ldots + (x_n - y_n)^2\right)^{1/2}.$$



- The set $BC(I, \mathbb{R})$ of **bounded and continuous functions** on an interval (open or closed) when endowed with the **supremum distance**:

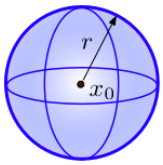
$$d_\infty(f, g) = \sup_{x \in I} |f(x) - g(x)|.$$

N.b. A set may be endowed with different metrics; the pair (X, d) defines a metric space.

Balls and neighborhoods

Let (X, d) be a metric space with distance $d\colon X \times X \to [0, \infty)$.

Two important concepts are the **ball of radius** $r > 0$ **centered at** $x_0 \in X$,



$$B_r(x_0) := \{x \in X\colon d(x, x_0) < r\},$$

and the **sphere of radius** $r > 0$ **centered at** $x_0 \in X$,

$$S_r(x_0) := \{x \in X\colon d(x, x_0) = r\}.$$

Ex.

- The ball of radius 2 centered at $(1, 0)$ in Euclidean space $\mathbb{R}^2$:

$$B_2((1, 0)) = \{(x, y) \in \mathbb{R}^2\colon (x - 1)^2 + y^2 < 4\}.$$

- The ball of radius 1 centered at 0 in $(BC([0, 1], \mathbb{R}), d_\infty)$ consists of all functions whose graph $y = f(x)$ lies strictly between the lines $y = \pm 1$.
- **Sequence spaces** are spaces in which each element

$$x = \{x_n\}_{n\in\mathbb{N}} = (x_1, x_2, \ldots)$$

is a sequence (usually of real or complex numbers). Let l_∞ be the space of sequences $x\colon \mathbb{N} \to \mathbb{R}$ for which

$$\sup_{n\in\mathbb{N}} |x_n| < \infty.$$

This space endowed with the distance

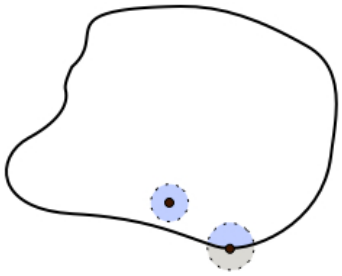
$$d_\infty(x, y) = \sup_n |x_n - y_n|$$

is a metric space.
Let 0 denote the zero sequence $(0, 0, 0...)$ and $x^{(0)} = (1/2, 2/3, 3/4, \ldots)$ Then $d_\infty(0, x^{(0)}) = 1$ since $\sup_{n\in\mathbb{N}} |\frac{n}{n+1}| = 1$, thus $x^{(0)} \in S_1(0)$.

Let (X, d) be a metric space and $x \in X$. A set $N \subset X$ is called a **neighborhood** of x if there exists $r > 0$ such that $B_r(x) \subset N$.

Interior points and boundary points

Let (X, d) be a metric space with distance $d\colon X \times X \to [0, \infty)$.



- A point $x_0 \in D \subset X$ is called an **interior point in D** if there is a ball centered at x_0 that lies entirely in D,

$$x_0 \text{ interior point} \quad \overset{\text{def}}{\iff} \quad \exists\, \varepsilon > 0; \qquad B_\varepsilon(x_0) \subset D.$$

- A point $x_0 \in X$ is called a **boundary point of D** if any ball centered at x_0 has non-empty intersections with both D and its complement,

$$x_0 \text{ boundary point} \quad \overset{\text{def}}{\iff} \quad \forall\, \varepsilon > 0 \quad \exists\, x, y \in B_\varepsilon(x_0); \quad x \in D,\ y \in X \setminus D.$$

- The set of interior points in D constitutes its **interior**, $\text{int}(D)$, and the set of boundary points its **boundary**, ∂D. D is said to be **open** if any point in D is an interior point and it is **closed** if its boundary ∂D is contained in D; the **closure of D** is the union of D and its boundary:

$$\overline{D} := D \cup \partial D.$$

Alternative notations for the closure of D in X include $\overline{D}^{X}$, $\operatorname{clos}(D)$ and $\operatorname{clos}(D; X)$.[1]

Ex.

- In $\mathbb{R}$ with the usual distance $d(x, y) = |x - y|$, the interval $(0, 1)$ is open, $[0, 1)$ neither open nor closed, and $[0, 1]$ closed.
- The set

$$D := \{(x, y) \in \mathbb{R}^2 \colon x > 0, y \geq 0\}$$

is neither closed nor open in Euclidean space $\mathbb{R}^2$, since its boundary contains both points $(x, 0)$, $x > 0$, in D and points $(0, y)$, $y \geq 0$, not in D. The interior and closure of D are

$$\operatorname{int}(D) = \{(x, y) \in \mathbb{R}^2 \colon x > 0, y > 0\}, \quad \overline{D} = \{(x, y) \in \mathbb{R}^2 \colon x \geq 0, y \geq 0\}.$$

- An entire metric space is both open and closed (its boundary is empty).
- For a general metric space, the **closed ball**

$$\tilde{B}_r(x_0) := \{x \in X \colon d(x, x_0) \leq r\}$$

may be larger than the closure of a ball, $\overline{B_r(x_0)}$. If we let X be a space with the discrete metric,

$$\begin{cases} d(x, x) & = 0, \\ d(x, y) & = 1, \quad x \neq y. \end{cases}$$

Then

$$B_1(x_0) = \{x_0\}, \quad \text{so that} \quad \overline{B_1(x_0)} = \overline{\{x_0\}} = \{x_0\}, \quad \text{but} \quad \tilde{B}_1(x_0) = X.$$

℘ **(Open) balls are open**

Let (X, d) be a metric space, x_0 a point in X, and $r > 0$. Then $B_r(x_0)$ is open in X with respect to the metric d.

Proof

Pick $x \in B_r(x_0)$. Then

$$\begin{aligned} d(x, x_0) < r \quad &\Longrightarrow \quad \exists\, \varepsilon > 0; \quad d(x, x_0) < r - \varepsilon \\ &\Longrightarrow \quad d(y, x) < \varepsilon \quad \text{implies} \quad d(y, x_0) \leq d(y, x) + d(x, x_0) < \varepsilon + (r - \varepsilon) = r. \end{aligned}$$

This means: $y \in B_r(x_0)$ if $y \in B_\varepsilon(x)$, i.e. $B_\varepsilon(x) \subset B_r(x_0)$.

2.2 Limits

Let (X, d) be a metric space with distance $d \colon X \times X \to [0, \infty)$.

Sequential limits

A sequence $\{x_n\}_{n \in \mathbb{N}} \subset X$ is said to **converge towards** $x_0 \in X$ if for any $\varepsilon > 0$ there is a natural number n_ε with the property that $x_n \in B_\varepsilon(x_0)$ for all $n \geq n_\varepsilon$:

$$\lim_{n \to \infty} x_n = x_0 \quad \overset{\text{def}}{\Longleftrightarrow} \quad \forall\, \varepsilon > 0 \quad \exists\, n_\varepsilon \in \mathbb{N}; \qquad x_n \in B_\varepsilon(x_0) \quad \text{for} \quad n \geq n_\varepsilon.$$

[1] An alternative to this approach is to take closed sets as complements of open sets. These two definitions, however, are completely equivalent.

We then say that x_n tends to x_0 as n tends to infinity, written

$$x_n \to x_0 \quad (\text{as} \quad n \to \infty), \qquad \text{or} \qquad x_n \overset{n\to\infty}{\to} x_0.$$

The point x_0 is called the **limit** of the sequence $\{x_n\}_{n\in\mathbb{N}}$.

℘ **Sequential limits are zero limits for the distance function**
Since $\{d(x_n, x_0)\}_{n\in\mathbb{N}}$ is a sequence in $\mathbb{R}$ it is easily verified that

$$x_n \to x_0 \quad \Longleftrightarrow \quad d(x_n, x_0) \to 0.$$

Proof

$$\begin{aligned}
d(x_n, x_0) \to 0 \quad &\Longleftrightarrow \quad \forall \varepsilon > 0 \quad \exists\, n_\varepsilon; \quad d(x_n, x_0) \in B_\varepsilon(0) \qquad (\text{in } \mathbb{R}) \quad \text{for} \quad n \geq n_\varepsilon \\
&\Longleftrightarrow \quad \forall \varepsilon > 0 \quad \exists\, n_\varepsilon; \quad d(x_n, x_0) < \varepsilon \qquad\qquad\quad \text{for} \quad n \geq n_\varepsilon \\
&\Longleftrightarrow \quad \forall \varepsilon > 0 \quad \exists\, n_\varepsilon; \quad x_n \in B_\varepsilon(x_0) \qquad (\text{in } X) \quad \text{for} \quad n \geq n_\varepsilon \\
&\Longleftrightarrow \quad x_n \to x_0.
\end{aligned}$$

℘ **Limits are unique**
If $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} x_n = y$, then $x = y$.

Proof
In view of the assumptions, and using the triangle inequality,

$$0 \leq d(x, y) \leq d(x, x_n) + d(x_n, y) \to 0 \quad \text{as} \quad n \to \infty.$$

Then $d(x, y) = 0$ implies $x = y$ by the axioms of a metric space.

Ex. The sequence

$$x_1 = 0.9, \quad x_2 = 0.99, \quad x_3 = 0.999, \quad \text{and so forth,}$$

converges towards $0.999\ldots$ in $\mathbb{R}$, but also towards 1. Hence

$$0.999\ldots = 1.$$

Relationship between limits and closures

℘ **Closures are the total of sequential limits of interior points**
By comparing the definitions of boundary and interior points with that of a sequential limit, one obtains that

$$\text{clos}(D; X) = \{x \in X : x = \lim_{n\to\infty} x_n \text{ for some sequence } \{x_n\}_{n\in\mathbb{N}}, x_n \in D \forall n\}$$

Proof
Assume that $x \in \overline{D}$. Then, according to the definitions of interior and boundary points, any small ball $B_{1/n}(x)$ contains a point $x_n \in D$. This means that $\{x_n\}_{n\in\mathbb{N}} \subset D$ converges to x.

Now, assume instead that there is a sequence

$$\{x_n\}_{n\in\mathbb{N}} \subset D \quad \text{with} \quad \lim_{n\to\infty} x_n = x \quad \text{in } X.$$

Then $d_X(x_n, x) \to 0$ as $n \to \infty$, so that

$$\forall\, \varepsilon > 0 \quad \exists\, x_{n_\varepsilon} \in B_\varepsilon(x).$$

Since $x_{n_\varepsilon} \in D$, either x is an interior point (for small ε there are only points from D in

$B_\varepsilon(x)$), or x is a boundary point ($B_\varepsilon(x)$ contains also points from the complement of D); in any case $x \in \overline{D}$.

Ex. Let $x^{(0)} = (1/2, 2/3, 3/4, \ldots) \in l_\infty$, then

$$x^{(0)} \in \overline{B_1(0)},$$

Consider the following sequence $x^{(1)} = (\frac{1}{2}, \frac{1}{2}, \ldots)$, $x^{(2)} = (\frac{1}{2}, \frac{2}{3}, \frac{2}{3}, \ldots)$, $x^{(3)} = (\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{3}{4}, \ldots)$, and so forth; the elements of which are all in $B_1(0)$ in l_∞. Then $x^{(m)} \overset{\text{in } l_\infty}{\to} x^{(0)}$ since

$$d_\infty(x^{(m)}, x^{(0)}) = \sup_{n \geq m} \left| \frac{n}{n+1} - \frac{m}{m+1} \right| = \left| 1 - \frac{m}{m+1} \right| = \frac{1}{m+1} \to 0 \quad \text{as} \quad n \to \infty.$$

Accumulation points

A concept related to convergence is that of an **accumulation point** of a sequence $x : \mathbb{N} \to X$:

$$x_0 \text{ accumulation point for } \{x_n\} \quad \overset{\text{def}}{\iff} \quad \exists \text{ a subsequence } \{x_{n_k}\}_{k \in \mathbb{N}}; \quad x_{n_k} \to x_0.$$

N.b. The limit of a sequence is always an accumulation point for that sequence, but a (non-convergent) sequence may have several, or no, accumulation points.

Ex. 0 is an accumulation point for $\{1/n\}_{n \in \mathbb{N}}$, but also for $\{1, 1, 2, \frac{1}{2}, 3, \frac{1}{3}, 4, \frac{1}{4}, \ldots\}$.

Bolzano-Weierstrass theorem

$\wp$ **Any bounded sequence in $\mathbb{R}^m$ has an accumulation point.**
Let $\{x_n\}$ be a sequence in $\mathbb{R}^m$ such that $\sup_n d(x_n, 0) \leq B$ for some $B \in \mathbb{R}$, where $d(x, y) = |x - y|$ is the Euclidean distance. Then $\{x_n\}$ has at least one accumulation point.

Proof

Assume first that $m = 1$. An index l is called a peak for $\{x_n\}$ if $x_l > x_n$ for all $n > l$. If the sequence has infinitely many peaks l_k then x_{l_k} form a decreasing sequence. It the sequence has only finitely many peaks then starting from n_1 that is larger than the largest peak we can choose a sequence $\{x_{n_k}\}$ by induction such that $x_{n_k} \leq x_{n_{k+1}}$. Hence the sequence $\{x_n\}$ contains a monotone subsequence.

Assume that $\{x_{n_k}\}$ is increasing and let $x_0 = \sup_k x_{n_k}$. Then $x_0 \leq B < +\infty$ since $|x_n| \leq B$. If $x_0 - x_{n_k} > \varepsilon$ for some $\varepsilon > 0$ and infinitely many k then $x_0 - x_{n_k} > \varepsilon$ for all k since $\{x_{n_k}\}$ is increasing, hence $x_0 - \varepsilon$ is an upper bound for $\{x_{n_k}\}$, but $x_0 - \varepsilon < x_0$ it contradicts the definition of the supremum. Thus for any $\varepsilon > 0$ there are only finitely many k such that $x_0 - x_{n_k} > \varepsilon$ and $x_{n_k} \to x_0$ as $k \to \infty$ and $\{x_n\}$ has an accumulation point. The same argument shows that if $\{x_{n_k}$ is decreasing then it converges to its infimum.

For $m \geq 2$, first, there is a subsequences with convergent first coordinate, from this subsequence one can choose a new subsequence with convergent second coordinate and so on. After m steps the resulting subsequence is convergent in $\mathbb{R}^m$.

A set K in a metric space X is called **compact** if every sequence in K contains a subsequence that converges to a point in K.[1] The Bolzano-Weierstrass theorem implies that a bounded and closed subset of $\mathbb{R}^m$ is compact. The converse is simple, if a subset of $\mathbb{R}^n$ is compact then it is bounded and closed. This characterization does not hold in general metric spaces.

[1] This is a definition of a sequentially compact set and a different definition of compactness is usually used for topological spaces, however for metric spaces the notions coincide.

2.3 Completeness

Cauchy sequences

A sequence $\{x_n\}$ in a metric space (X, d) is a **Cauchy sequence** if the distance between its members tends to zero:

$$\{x_n\} \text{ is a Cauchy sequence} \quad \overset{\text{def}}{\iff} \quad d(x_n, x_m) \to 0 \quad \text{as} \quad m, n \to \infty.$$

Equivalently,

$$\forall\, \varepsilon > 0 \quad \exists\, n_\varepsilon; \qquad d(x_m, x_n) < \varepsilon \quad \text{whenever} \quad m, n \geq n_\varepsilon.$$

Ex.

- The sequence $\{x_n\}_{n\geq 1}$ of rational numbers $x_n = \sum_{k=0}^{n} \frac{1}{k!}$ is a Cauchy sequence with respect to the distance $d(x, y) = |x - y|$. For each $m \geq n \geq 1$,

$$\begin{aligned} |x_m - x_n| &= \sum_{k=n+1}^{m} \frac{1}{k!} \leq \frac{1}{(n+1)!} \sum_{k=0}^{m-(n+1)} \frac{1}{(n+1)^k} \\ &= \frac{1}{(n+1)!} \frac{1 - (\frac{1}{n+1})^{(m-n)}}{1 - \frac{1}{n+1}} \overset{n\geq 1}{\leq} \frac{2}{(n+1)!} \to 0 \quad \text{as} \quad m \geq n \to \infty. \end{aligned}$$

  Note, however, that $\lim_{n\to\infty} x_n = e \notin \mathbb{Q}$.
- The sequence $\{x_n\}_{n\geq 1}$ of functions $x_n : t \mapsto \sum_{k=0}^{n} \frac{t^k}{k!}$ is Cauchy in $BC([0, 1], \mathbb{R})$. For each $m \geq n \geq 1$,

$$d_\infty(x_m, x_n) = \sup_{t\in[0,1]} \Big| \sum_{k=n+1}^{m} \frac{t^k}{k!} \Big| \leq \sum_{k=n+1}^{m} \frac{1}{k!} \to 0 \quad \text{as} \quad m \geq n \to \infty.$$

  In this case $\lim_{n\to\infty} x_n = [t \mapsto e^t] \in BC([0, 1], \mathbb{R})$.[1]

℘ **Convergent sequences are Cauchy sequences**

In any metric space

$$x_n \to x \ \text{ as } \ n \to \infty \qquad \text{implies} \qquad d(x_m, x_n) \to 0 \ \text{ as } \ m, n \to \infty.$$

N.b. The opposite is not true in general.

Proof

Let $\varepsilon > 0$. Since $x_n \to x$ there exists n_ε such that

$$d(x, x_n) < \varepsilon/2 \quad \text{for} \quad n \geq n_\varepsilon.$$

Hence

$$d(x_m, x_n) \leq d(x_m, x) + d(x, x_n) < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

for $m, n \geq n_\varepsilon$.

℘ **Cauchy sequences are bounded**

If (X, d) is a metric space, $\{x_n\}_n \subset X$ is a Cauchy sequence and $y \in X$, then

$$\sup_{n\geq 1} d(x_n, y) < B \quad \text{for some} \quad B \in \mathbb{R}.$$

[1] The exponential function $t \mapsto e^t$ is often expressed as exp to separate it from the real number $e = \exp(1)$.

Proof

According to the definition of a Cauchy sequence, there exists n_1 such that

$$d(x_n, x_m) < 1 \quad \text{for} \quad m, n \geq n_1.$$

If $n \leq n_1$,

$$d(x_n, y) \leq \max_{1 \leq n \leq n_1} d(x_n, y) =: \tilde{B},$$

and, if $n \geq n_1$,

$$d(x_n, y) \leq d(x_n, x_{n_1}) + d(x_{n_1}, y) < 1 + d(x_{n_1}, y).$$

Hence, $d(x_n, y) \leq B := \max\{\tilde{B}, 1 + d(x_{n_1}, y)\}$ for all $n \in \mathbb{N}$.

Complete metric spaces

A metric space in which every Cauchy sequence converges is called **complete**.

Ex.

- $\mathbb{Q}$ is not complete with respect to the metric $d(x, y) = |x - y|$.
- Any set with discrete metric is complete.

$\wp$ $\mathbb{R}^m$ is complete

The Euclidean space $\mathbb{R}^m$ is complete with respect to the metric $d(x, y) = |x - y|$.

Proof

Let $\{x_n\}$ be a Cauchy sequence in $\mathbb{R}^m$. Then it is bounded and by the Bolzano-Weierstrass theorem it has a convergent subsequence $x_{n_k} \to x_0$ as $k \to \infty$. Let $\varepsilon > 0$ there exists j such that $d(x_n, x_l) < \varepsilon/2$ when $n, l > j$ and there exists k_0 such that $d(x_{n_k}, x_0) < \varepsilon/2$ when $k \geq k_0$. Let $k > k_0$ be such that $n_k > j$. Then

$$d(x_l, x_0) \leq d(x_l, x_{n_k}) + d(x_{n_k}, x_0) < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

for any $l > j$.

$\wp$ Subsets of complete metric spaces are complete if and only if they are closed

Let $M \subset X$ be a subset of a complete metric space (X, d), i.e. $(M, d) \subset (X, d)$. Then

$$M \text{ complete} \quad \Longleftrightarrow \quad M \text{ closed}.$$

Proof

Assume that M is complete. Closedness means

$$M \ni x_n \to x \in X \quad \Longrightarrow \quad x \in M.$$

So assume that $\{x_n\} \subset M$ converges towards $x \in X$. Since any convergent sequence is Cauchy, $\{x_n\} \subset M$ is Cauchy. But since M is complete, $\{x_n\}$ converges to an element $y \in M$. By uniqueness of limits, $y = x$. Thus M is closed.

Contrariwise, assume that M is closed and let $\{x_n\} \subset M$ be a Cauchy sequence. Recall that $(M, d) \subset (X, d)$ carry the induced metric d. Thus

$$\{x_n\} \text{ Cauchy in } M \implies \{x_n\} \text{ Cauchy in } X \overset{X \text{ complete}}{\Longrightarrow} x_n \to x \in X \overset{M \text{ closed}}{\Longrightarrow} x \in M.$$

[1] Note that $|z_j|^2 = a_j^2 + b_j^2$ for complex numbers $x_j = a_j + ib_j$.

Thus M is complete.

Ex.

- A subset of $\mathbb{R}^n$ is complete if and only if it is closed, for example the closed interval $[0,1]$ is complete and the open ball $\{x \in \mathbb{R}^2 : |x| < 1\}$ is not.
- The sequence space l_2 is defined as the set of all sequences $x = \{x_n\}$ in $\mathbb{R}$ (or $\mathbb{C}$) such that $\sum_n |x_n|^2 < +\infty$. It is a metric space with respect to the distance

$$d_2(x,y) = \left(\sum_n |x_n - y_n|^2 \right)^{1/2}.$$

(The triangle inequality follows from the Cauchy-Schwarz inequality which will be proved later). This space is complete (also to be proved later).
The Euclidean spaces $\mathbb{R}^n$ can be viewed as closed subset of l_2:

$$\mathbb{R}^n = \{(x_1, \ldots, x_n, 0, 0 \ldots) \in l_2\}.$$

2.4 Important examples of complete metric spaces

In this section we prove that sequence spaces l_2, l_∞ and function space $BC(I, \mathbb{R})$ are complete. The proofs work for complex valued sequences and functions.

℘ **The space of square-summable sequences is complete**

The space of square-summable (complex or real) sequences is a Banach space with respect to the norm $\|x\|_{l_2} = (\sum_{j \geq 1} |x_j|^2)^{1/2}$.

Proof

This time we pursue the proof for complex-valued sequences (it is no different from the proof for real-valued sequences).

Pointwise convergence: Take a Cauchy sequence $\{x_n\}_n$ in l_2, with $x_n = (x_n(1), x_n(2), \ldots)$. Then

$$|x_n(j) - x_m(j)| \leq \Big(\sum_{j=1}^{\infty} |x_n(j) - x_m(j)|^2 \Big)^{1/2} = d_2(x_n, x_m) \to 0 \quad \text{as} \quad m, n \to \infty.$$

Thus, for each $j \in \mathbb{N}$, $\{x_n(j)\}_n$ is a Cauchy sequence in $\mathbb{C}$, and thus it converges:

$$\forall\, j \in \mathbb{N} \quad \exists\, x_0(j) = \lim_{n \to \infty} x_n(j) \in \mathbb{C}.$$

Let $x_0 := (x_0(1), x_0(2), \ldots)$.

Boundedness: For any $j \in \mathbb{N}$ there exists $n_j \in \mathbb{N}$ such that

$$|x_0(j) - x_n(j)|^2 < \frac{1}{2^j} \quad \text{for } n \geq n_j.$$

For any finite $N \in \mathbb{N}$, choose $n \geq \max_{1\leq j\leq N} n_j$. Then

$$\Big(\sum_{j=1}^{N} |x_0(j)|^2\Big)^{1/2} \leq \Big(\sum_{j=1}^{N} |x_0(j) - x_n(j)|^2\Big)^{1/2} + \Big(\sum_{j=1}^{N} |x_n(j)|^2\Big)^{1/2}$$
$$< \Big(\sum_{j=1}^{N} \frac{1}{2^j}\Big)^{1/2} + d_2(x_n, 0) \leq\leq 1 + \sup_{n\in\mathbb{N}} d_2(x_n, 0) < c,$$

since Cauchy sequences are bounded. The right-hand side is independent of N, so we may now let $N \to \infty$, yielding that

$$\sum_j |x_0(j)|^2 = d_2(x_0, 0) \leq c, \quad \text{whence} \quad x_0 \in l_2.$$

Convergence in metric: Let $\varepsilon > 0$. As above, we can find n_j such that

$$|x_0(j) - x_m(j)|^2 < \frac{\varepsilon/2}{2^j} \quad \text{for } m \geq n_j,$$

and

$$\sum_{j=1}^{N} |x_0(j) - x_m(j)|^2 < \sum_{j=1}^{N} \frac{\varepsilon/2}{2^j} \leq \varepsilon/2 \quad \text{for} \quad m \geq \max_{1\leq j\leq N} n_j.$$

Using this,

$$\Big(\sum_{j=1}^{N} |x_0(j) - x_n(j)|^2\Big)^{1/2} \leq \Big(\sum_{j=1}^{N} |x_0(j) - x_m(j)|^2\Big)^{1/2} + \Big(\sum_{j=1}^{N} |x_m(j) - x_n(j)|^2\Big)^{1/2}$$
$$\leq \varepsilon/2 + d_2(x_n, x_m).$$

Since $\{x_n\}_n$ is a Cauchy sequence in l_2 there exists $n_\varepsilon \in \mathbb{N}$ such that

$$d_2(x_n, x_m) < \varepsilon/2 \quad \text{for} \quad m, n \geq n_\varepsilon.$$

Select m such that this holds (for example, $m \geq n_\varepsilon + \max_{1\leq j\leq N} n_j$). Then

$$\left(\sum_{j=1}^{N} |x_0(j) - x_n(j)|^2\right)^{1/2} < \varepsilon \quad \text{for} \quad n \geq n_\varepsilon.$$

Since n_ε does not depend on N, we may let $N \to \infty$, to obtain that

$$d(x_0, x_n) < \varepsilon \quad \text{for} \quad n \geq n_\varepsilon.$$

Hence, $x_n \overset{\text{in } l_2}{\to} x_0$, and l_2 is complete.

A similar argument shows that l_∞ is complete.

℘ **BC is complete**
Let $I \subset \mathbb{R}$ be a non-empty interval. Then $BC(I, \mathbb{R})$ and $BC(I, \mathbb{C})$ are complete metric spaces.

Proof
The proof is the same for $\mathbb{R}$ and $\mathbb{C}$.

Let $\{x_n\}_n$ be a Cauchy sequence in $BC(I,\mathbb{R})$. We want to prove that it converges to a limit function $x_0 \in BC(I,\mathbb{R})$.

Pointwise convergence: For any $t \in I$,

$$|x_n(t) - x_m(t)| \leq \sup_{t\in I} |x_n(t) - x_m(t)| = d_\infty(x_n, x_m).$$

Thus

$$\{x_n\}_n \text{ Cauchy in } BC(I,\mathbb{R}) \quad \Longrightarrow \quad \{x_n(t)\}_n \text{ Cauchy in } \mathbb{R}.$$

$\mathbb{R}$ being complete, there exists a limit in $\mathbb{R}$:

$$\forall\, t \in I \quad \exists\, x_0(t) := \lim_{n\to\infty} x_n(t) \quad \text{ in } \mathbb{R}.$$

Define a function x_0 by $x_0 := [t \mapsto x_0(t)]$.

Boundedness: For each $t \in I$ there exists $n_t \in \mathbb{N}$ such that $|x_0(t) - x_{n_t}(t)| < \varepsilon$. Thus

$$|x_0(t)| \leq |x_0(t) - x_{n_t}(t)| + |x_{n_t}(t)| < \varepsilon + \sup_{s\in I} |x_{n_t}(s)| \leq \varepsilon + \sup_{n\in\mathbb{N}} d_\infty(x_n, 0) < C,$$

since Cauchy sequences are bounded. Taking the supremum over all $t \in I$ yields that

$$\sup_{t\in I} |x_0(t)| < C.$$

Convergence in metric: A similar argument shows that $\sup_{t\in I} |x_n - x_0|$ goes to 0 as $n \to \infty$. Let $\varepsilon > 0$. For any $t \in I$ there exists $m_t \in \mathbb{N}$ such that

$$|x_0(t) - x_m(t)| < \frac{\varepsilon}{2} \quad \text{ for } \quad m \geq m_t.$$

Also, there exists $n_\varepsilon \in \mathbb{N}$ such that

$$d_\infty(x_n, x_m) < \frac{\varepsilon}{2} \quad \text{ for } \quad m, n \geq n_\varepsilon.$$

Choose $m \geq \max\{m_t, n_\varepsilon\}$. Then

$$\begin{aligned} |x_n(t) - x_0(t)| &\leq |x_n(t) - x_m(t)| + |x_m(t) - x_0(t)| \leq d_\infty(x_n, x_m) + |x_m(t) - x_0(t)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \text{ for } \quad n \geq n_\varepsilon. \end{aligned}$$

Taking the supremum over $t \in I$ yields that

$$\sup_{t\in I} |x_n(t) - x_0(t)| \leq \varepsilon \quad \text{ for } \quad n \geq n_\varepsilon.$$

To show that $x_n \to x_0$ in $BC(I,\mathbb{R})$ one needs to check that $x_0 \in BC(I,\mathbb{R})$.

Continuity: To prove that x_0 is continuous, pick $t \in I$ and let $\varepsilon > 0$. Since $\sup_{t\in I} |x_n(t) - x_0(t)| \to 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\sup_{t\in I} |x_n(t) - x_0(t)| < \frac{\varepsilon}{3} \quad \text{ for } \quad n \geq n_\varepsilon.$$

Fix such an n. Since x_n is continuous at t, there exists $\delta := \delta(n, \varepsilon) > 0$ with

$$|x_n(s) - x_n(t)| < \frac{\varepsilon}{3} \quad \text{for} \quad |s - t| < \delta.$$

All taken together,

$$\begin{aligned} |x_0(s) - x_0(t)| &< |x_0(s) - x_n(s)| + |x_n(s) - x_n(t)| + |x_n(t) - x_0(t)| \\ &\leq \sup_{p \in I} |x_0(p) - x_n(p)| + |x_n(s) - x_n(t)| + \sup_{p \in I} |x_0(p) - x_n(p)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \quad \text{for} \quad |s - t| < \delta. \end{aligned}$$

Hence, x_0 is continuous at t. This shows that every Cauchy sequence $\{x_n\}_n \subset BC(I, \mathbb{R})$ converges (in $BC(I, \mathbb{R})$) towards an element $x_0 \in BC(I, \mathbb{R})$. Thus $BC(I, \mathbb{R})$ is complete.

The same proof shows that $BC(I, \mathbb{R}^n)$ is complete.

2.5 Functions between metric spaces

Continuity

Let (X, d_X) and (Y, d_Y) be metric spaces, and $f\colon X \to Y$ a function between them. We say that $f(x)$ **converges to** y_0 **in** Y **as** x **converges to** x_0 **in** X if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $f(x) \in B_\varepsilon(y_0)$ when $x \in B_\delta(x_0)$:

$$\lim_{x \to x_0} f(x) = y_0 \quad \overset{\text{def}}{\iff} \quad \forall\, \varepsilon > 0 \quad \exists\, \delta > 0; \qquad [d_X(x, x_0) < \delta \quad \implies \quad d_Y(f(x), y_0) < \varepsilon].$$

Equivalent ways of writing this are

$$f(x) \to y_0 \quad \text{as} \quad x \to x_0, \qquad \text{and} \qquad f(x) \overset{x \to x_0}{\to} y_0.$$

A function f satisfying this is said to be **continuous** at the point x_0. It is continuous on a set D if it continuous at all points $x_0 \in D$, and simply continuous if its continuous on all of its domain.

Ex. The function

$$f\colon x \mapsto \frac{\sin(x)}{x}, \qquad x \in \mathbb{R} \setminus \{0\},$$

may be extended to a bounded and continuous function $\mathbb{R} \to \mathbb{R}$, since

$$f(x) \overset{\text{in } \mathbb{R}}{\to} 1 \qquad \text{as} \qquad x \overset{\text{in } \mathbb{R}}{\to} 0.$$

℘ **In metric spaces continuous and sequential limits agree**

$$(i) \quad \lim_{x \to x_0} f(x) = y \quad \iff \quad (ii) \quad \lim_{n \to \infty} f(x_n) = y \text{ for any sequence such that } \lim_{n \to \infty} x_n = x_0.$$

Proof

Assume that (i) holds, i.e.,

$$d_Y(f(x), y) < \varepsilon \quad \text{for} \quad d_X(x, x_0) < \delta.$$

For any sequence $\{x_n\}_{n \in \mathbb{N}} \subset X$ with $\lim_{n \to \infty} x_n = x_0$, there exists $N \in \mathbb{N}$ with

$$d_X(x_n, x_0) < \delta \quad \text{for} \quad n \geq N.$$

Thus, according to (i),

$$d_Y(f(x_n), y) < \varepsilon \quad \text{for} \quad n \geq N.$$

This shows that (i) is sufficient for (ii) to hold.

Now assume that (i) does not hold, i.e., there is an $\varepsilon > 0$ such that *for any* $\delta > 0$ there exists x_δ with

$$d_X(x_\delta, x_0) < \delta \quad \text{while} \quad d_Y(f(x_\delta), y) \geq \varepsilon.$$

Thus, any sequence of $\delta_n \overset{n\to\infty}{\to} 0$ yields a sequence of numbers $x_n := x_{\delta_n}$ with

$$x_n \to x_0 \text{ as } n \to \infty \quad \text{while} \quad d_Y(f(x_n), y) \geq \varepsilon.$$

This violates (ii) and shows that (i) is necessary for (ii) to hold.

Ex.

- The function

$$f\colon x \mapsto \sin\left(\frac{1}{x}\right), \qquad x \in \mathbb{R} \setminus \{0\},$$

is continuous on $\mathbb{R}\setminus\{0\}$ but it can not be extended to a bounded and continuous function $\mathbb{R} \to \mathbb{R}$, since

$$f(1/\pi k) \to 0, \quad f(2/(4k+1)\pi) \to 1.$$

- The sequence of functions given by

$$f_0(x) = 1, \quad f_1(x) = 1 - \frac{x^2}{3!}, \quad f_n(x) = \sum_{j=0}^{n} \frac{(-1)^j x^{2j}}{(2j+1)!} \quad n \in \mathbb{N},$$

converges in $BC([0,1], \mathbb{R})$. Namely, let $f(x) = \frac{\sin(x)}{x}$, extended to a bounded and continuous function on $\mathbb{R}$ as in the preceding example. Since

$$\sin x \overset{\text{Taylor}}{=} x - \frac{x^3}{3!} + \ldots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} \pm \frac{\cos(\xi)\, x^{2n+3}}{(2n+3)!}, \qquad 0 < |\xi| < |x|,$$

we have

$$\frac{\sin x}{x} = \sum_{j=0}^{n} \frac{(-1)^j x^{2j}}{(2j+1)!} \pm \frac{\cos(\xi)\, x^{2n+2}}{(2n+3)!}, \qquad 0 < |\xi| < |x|,$$

so that

$$\begin{aligned} d_\infty(f_n, f) &= \sup_{x\in[0,1]} \left| \sum_{j=0}^{n} \frac{(-1)^j x^{2j}}{(2j+1)!} - \frac{\sin x}{x} \right| \\ &\leq \sup_{x,\xi\in[0,1]} \left| \frac{\cos(\xi)\, x^{2n+2}}{(2n+3)!} \right| = \frac{1}{(2n+3)!} \to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Continuous functions on compact sets

Let $f : X \to \mathbb{R}$ be a function from a metric space X to the space of the real number with usual metric and let K be compact subset of X.

℘ **Extremal value theorem**
If f is continuous on K then it is bounded and attains its maximum value, i.e., there exists $x_0 \in K$ such that $f(x_0) = \sup_{x\in K} f(x)$.

Proof
Suppose first that f is unbounded and $\sup_{x\in K} = +\infty$. Then there is a sequence $x_n \in K$ such that $f(x_n) > n$. Since K is compact there exists a convergent subsequence of this sequence, $x_{n_k} \to x_0, k \to \infty$. Then the continuity of f implies $f(x_{n_k}) \to f(x_0)$ which contradicts the choice $f(x_{n_k}) > n_k$. Similarly $\inf_{x\in k} f(x) > -\infty$ and f is bounded on K.

Now let $s = \sup_{x\in K} f(x)$, then there exists a sequence x_n such that $f(x_n) > s - 1/n$. Once again there is a convergent subsequence x_{n_k}, $x_{n_k} \to x_0, k \to \infty$. By the continuity of f,

$$f(x_0) = \lim_{k\to\infty} f(x_{n_k}) \geq s,$$

but $s = \sup_{x\in K} f(X) \geq f(x_0)$. Thus $f(x_0) = \sup_{x\in K} f(x)$.

For example any continuous function on a closed interval $[a, b]$ is bounded and attains its maximum and minimum values.

Let now $f : x \to Y$ be a function between metric spaces (X, d_X) and (Y, d_Y), remind that f is continuous on X if

$$\forall x \in X \forall \varepsilon > 0 \exists \delta > 0; \qquad [d_X(x, x') < \delta \quad \Longrightarrow \quad d_Y(f(x), f(x')) < \varepsilon].$$

A function $f : X \to Y$ is called **uniformly continuous** on X if

$$\forall \varepsilon > 0 \exists \delta > 0; \qquad [d_X(x, x') < \delta \quad \Longrightarrow \quad d_Y(f(x), f(x')) < \varepsilon].$$

An uniformly continuous function is continuous but not all continuous functions are uniformly continuous.

Ex.
- The function

$$f\colon x \mapsto \frac{1}{x}, \quad x \in (0, 1),$$

  is continuous on $(0, 1)$ but not uniformly continuous.
- The function $f(x) = \sin x$ is uniformly continuous on $\mathbb{R}$.
- The function $f(x) = x$ is uniformly continuous on $\mathbb{R}$ but $f(x) = x^2$ is not.
- A function $f : X \to Y$ is **uniformly Lipschitz continuous** if there exists a constant L such that

$$d_Y(f(x_1), f(x_2)) \leq L d_X(x_1, x_2), \qquad \text{for any } x_1, x_2 \in X.$$

  Uniformly Lipschitz continuous functions are uniformly continuous but not all uniformly continuous functions are Lipschitz.

℘ **Continuous functions are uniformly continuous on compact sets**
If $f : X \to Y$ is continuous and X is compact, then f is uniformly continuous.

The proof once again follows from the definition of a compact metric space, we omit it here.

2.6 The Banach fixed-point theorem

Contractions

Let (X, d) be metric space. A mapping $T : X \to X$ is called a **contraction** if there exists $\lambda < 1$ such that

$$d(T(x), T(y)) \leq \lambda\, d(x, y), \qquad \text{for all} \quad x, y \in X.$$

In particular, contractions are Lipschitz functions and hence are continuous.

Ex. The following functions are contractions on $\mathbb{R}$ and $\mathbb{R}^2$ with usual Euclidean norms

- $T : \mathbb{R} \to \mathbb{R}$, $T(x) = qx$ is a contraction when $|q| < 1$
- If $f : \mathbb{R} \to \mathbb{R}$ has a continuous derivative and $\sup_{\mathbb{R}} |f'| < 1$ then f is a contraction.
- $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 0.5 & 0 \\ 0 & 0.8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ is a contraction.

℘ **The Banach fixed-point theorem**

Let T be a contraction on a complete metric space (X, d) with $X \neq \emptyset$. Then there exists a unique $x \in X$ such that $T(x) = x$.

Proof (*if you are to learn one proof, this is the one*)

Existence of a candidate for x: Let $x_0 \in X$,

$$x_1 := T(x_0), \qquad x_{n+1} := T(x_n) = T^{n+1}(x_0), \quad n \in \mathbb{N}.$$

For $n > m \geq n_0$, we have that

$$\begin{aligned} d(x_n, x_m) &\overset{\Delta\text{-ineq.}}{\leq} \sum_{k=m+1}^{n} d(x_k, x_{k-1}) \overset{\text{def.}x_n}{=} \sum_{k=m+1}^{n} d\left(T^k(x_0), T^{k-1}(x_0)\right) \\ &\overset{\text{contr.}}{\leq} \sum_{k=m+1}^{n} \lambda^{k-1} d(x_1, x_0) = d(x_1, x_0)\, \lambda^m \sum_{k=0}^{n-m-1} \lambda^k \\ &\overset{\text{geom. series}}{=} d(x_1, x_0)\, \lambda^m \frac{1 - \lambda^{n-m}}{1 - \lambda} \leq \frac{\lambda^{n_0}}{1 - \lambda} d(x_1, x_0) \overset{n_0 \to \infty}{\to} 0. \end{aligned}$$

Thus $\{x_n\}_n$ is a Cauchy sequence. By assumption, (X, d) is complete, so there exists

$$x := \lim_{n \to \infty} x_n \in X.$$

x is a fixed point for T:

$$\begin{aligned} 0 \leq d(x, T(x)) &\leq d(x, x_n) + d(x_n, T(x_n)) + d(T(x_n), T(x)) \\ &\leq d(x, x_n) + d(x_n, x_{n+1}) + \lambda\, d(x_n, x) \\ &\leq \underbrace{d(x, x_n)}_{\to 0} + \underbrace{\lambda^n}_{\to 0} d(x_0, x_1) + \lambda \underbrace{d(x_n, x)}_{\to 0} \to 0, \qquad \text{as} \quad n \to \infty. \end{aligned}$$

That is: $x = T(x)$ is a fixed point for T.

Uniqueness: Assume that $y = T(y)$. Then

$$0 \leq d(x, y) = d(T(x), T(y)) \leq \lambda\, d(x, y) \quad \overset{\lambda \leq 1}{\Longrightarrow} \quad d(x, y) = 0 \quad \Longrightarrow \quad y = x.$$

N.b. The uniformity of the constant $\lambda < 1$ is important; it is not enough that $d(T(x), T(y)) < d(x, y)$ for each pair $(x, y) \in X \times X$.

Ex.

- Let $f : [a, b] \to [a, b]$ satisfy the Lipschitz condition $|f(x) - f(y)| \leq K|x - y|$ with $K < 1$. Then f is a contraction and by the Banach fixed point theorem $f(x) = x$ has a unique solution that can be found by **simple iterations**

$$x_{k+1} = f(x_k).$$

- Let now $F \in C^1([a, b], \mathbb{R})$ and suppose that we want to find its zero point (i.e., a solution of $F(x) = 0$). Assume that $0 \leq C_1 \leq F' \leq C_2$ on $[a, b]$ and $F(a) < 0 < F(b)$. Then F is increasing an has at exactly one zero point. We take

$$f(x) = x - cF(x)$$

and choose c such that f maps $[a, b]$ to $[a, b]$ and is a contraction. Then a zero point of F is a fixed point of f and we find it by iterations

$$x_{k+1} = x_k - cF(x_k).$$

- If F is as in the previous example and in addition $F \in C^2([a, b])$ then

$$g(x) = x - F(x)/F'(x)$$

is a contraction in some neighborhood of the zero point and one may use iterations of g to find it (**Newton's method**)

$$x_{k+1} = x_k - \frac{F(x_k)}{F'(x_k)}$$

with x_0 close enough to the fixed point.

Note that the Banach fixed point theorem is not a pure existence statement, it provides an iteration method $x_n \to x_{n+1}$ that converges to the fixed point x and gives the following estimate for convergence of this method,

$$d(x_n, x) \leq \frac{\lambda^n}{1 - \lambda} d(x_1, x_0).$$

In applications, one can construct the function T whose fixed point would satisfy a given equation (as it was demonstrated by examples above). Further, if T is already chosen, a metric on X and a subset of $Y \subset X$ on which T is considered could be adjusted to give in a contraction $T : Y \to Y$.

Ex.

- $T(x) = (1 + x^2)/3$ is a contraction on $[0, 1]$ but not on $[0, 2]$, the iterations converge to a solution of

$$x^2 - 3x + 1 = 0.$$

This equation has one solution on $[0, 1]$ and one on $[2, 3]$.

- $T(x) = \frac{x+1}{4} + \frac{2}{x}$ is a contraction on $[2, +\infty)$, the iterations converge to the positive solution of

$$3x^2 - x - 8 = 0.$$

In the examples below a simple function on $\mathbb{R}^n$ is considered in different metrics.

Ex. Let $A = \{a_{ij}\}$ be a square $n \times n$ matrix, it defines a function $T : x \mapsto Ax + x_0$ on $\mathbb{R}^n$.

- If $\mathbb{R}^n$ is endowed with the metric d_∞, then the map above is a contraction when

$$\max_j \sum_{k=1}^{n} |a_{jk}| < 1.$$

- If $\mathbb{R}^n$ is endowed with the metric d_1 then T is a contraction when

$$\max_k \sum_{j=1}^{n} |a_{jk}| < 1.$$

- For the standard d_2 metric, T is a contraction when

$$\sum_{j=1}^{n} \sum_{k=1}^{n} |a_{jk}|^2 < 1.$$

2.7 An application: existence theorems for ODE

Let $|\cdot|$ denote the Euclidean norm on $\mathbb{R}^n$.

Initial-value problems

Let (t_0, x_0) be a fixed point in an open subset $I \times U \subset \mathbb{R} \times \mathbb{R}^n$, and $f \in C(I \times U, \mathbb{R}^n)$ a continuous vector-valued function on this subset. The problem of finding $x \in C^1(J, U)$ such that

$$\dot{x}(t) = f(t, x(t)), \qquad x(t_0) = x_0, \qquad \text{(IVP)}$$

for some possibly smaller interval $J \subset I$ is called an **initial-value problem**. Here, $\dot{x} = \frac{d}{dt}x$.

℘ **Reformulation of real-valued ODEs as first-order systems**

Any ordinary differential equation

$$x^{(n)}(t) = g(t, x(t), \dot{x}(t), \ldots, x^{(n-1)}(t)),$$

with initial conditions

$$x(t_0) = x_1, \quad \dot{x}(t_0) = x_2, \quad \ldots, \quad x^{(n-1)}(t_0) = x_n,$$

and g continuous in some open set $I \times U \subset \mathbb{R} \times \mathbb{R}^n$ containing $(t_0, x_1, \ldots, x_n)$, can be reformulated in the form (IVP).

Proof

Let

$$y_0 := x, \quad y_1 := \dot{x}, \quad \ldots, \quad y_{n-1} := x^{(n-1)}.$$

Then

$$\begin{bmatrix} \dot{y}_0 \\ \vdots \\ \dot{y}_{n-2} \\ \dot{y}_{n-1} \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_{n-1} \\ g(t, y_0, \ldots, y_{n-1}) \end{bmatrix}$$

describes $y = (y_0, \ldots, y_{n-1})$ as a function $I \to U \subset \mathbb{R}^n$ for some interval $I \subset \mathbb{R}$; the function $f \in C(I \times U, \mathbb{R}^n)$ is the vector-valued function given by the right-hand side of this system. The initial condition is $y(t_0) = (x_1, \ldots, x_n)$.

Ex.

- The second-order ordinary differential equation

$$\ddot{x} + \sin(x) = 0, \qquad x(0) = 1, \quad \dot{x}(0) = 2,$$

is equivalent to the system

$$\begin{bmatrix} \dot{y}_0 \\ \dot{y}_1 \end{bmatrix} = \begin{bmatrix} y_1 \\ -\sin(y_0) \end{bmatrix} \qquad \text{with} \qquad \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}_{t=0} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

In this case

$$f\colon \mathbb{R}^2 \to \mathbb{R}^2, \qquad \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} \mapsto \begin{bmatrix} y_1 \\ -\sin(y_0) \end{bmatrix}$$

is independent of time.

℘ The Peano existence theorem

For any $(t_0, x_0) \in I \times U$ there exists $\varepsilon > 0$ such that the initial-value problem (IVP) has a solution defined for $|t - t_0| < \varepsilon$. The solution $x = x(\cdot; t_0, x_0) \in C^1(B_\varepsilon(t_0), U)$.

N.b. The Peano existence theorem guarantees the existence of (local) solutions, but not their uniqueness. We do not prove it here.

Ex.

- The initial-value problem

$$\begin{cases} \dot{x} = \frac{3}{2}x^{1/3}, & t \geq 0, \\ \dot{x} = 0, & t < 0, \end{cases} \qquad x(0) = 0,$$

has the trivial solution $x \equiv 0$, but also the ones given by

$$\begin{cases} x(t) = \pm t^{3/2}, & t \geq 0, \\ x(t) = 0, & t < 0. \end{cases}$$

To remedy this lack of uniqueness in Peano's theorem one needs the concept of Lipschitz continuity.

Lipschitz continuity

A continuous function $f \in C(I \times U, \mathbb{R}^n)$ is said to be locally **Lipschitz continuous** with respect to its second variable $x \in U$ if for any $(t_0, x_0) \in I \times U$ there exists $\varepsilon, L > 0$ with

$$|f(t, x) - f(t, y)| \leq L\,|x - y|, \qquad \text{for all} \quad (t, x), (t, y) \in B_\varepsilon(t_0, x_0).$$

If the Lipschitz constant L does not depend on the point (t_0, x_0), then the Lipschitz condition is said to be ***uniform*** (f is then uniformly Lipschitz continuous). *A locally Lipschitz continuous function is uniformly Lipschitz continuous on any compact set.*

N.b. Any continuously differentiable function is also locally Lipschitz continuous, and hence uniformly Lipschitz on any compact set.

Ex. Consider $f\colon \mathbb{R} \to \mathbb{R}$ (one spatial variable, no time).

- $x \mapsto \sin(x)$ is continuously differentiable. It is also uniformly Lipschitz, since
$$|\sin(x) - \sin(y)| \leq \max_{\xi \in \mathbb{R}} |\cos(\xi)||x - y|.$$
- $x \mapsto x^2$ is continuously differentiable. It is locally Lipschitz, since
$$|x^2 - y^2| = |x + y||x - y|.$$
- $x \mapsto |x|$ is not continuously differentiable. It is however (uniformly) Lipschitz, since
$$||x| - |y|| \leq |x - y|.$$
- $x \mapsto \sqrt{|x|}$ is continuous but not locally Lipschitz, since it cannot have a finite Lipschitz constant at $x_0 = 0$:
$$\frac{\sqrt{|x|}}{|x|} \to \infty \text{ as } x \to 0.$$

In particular, the examples above show that $C^1(\mathbb{R}) \subsetneq Lip(\mathbb{R}) \subsetneq C^0(\mathbb{R})$.[1]

An application of the Banach fixed-point theorem to ODE

Outline of the approach

To solve the initial-value problem (IVP) we shall reformulate it as

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds, \qquad x \in BC(I, U),$$

where the right-hand side defines a (not necessarily linear) mapping

$$T\colon BC(J, U) \to BC(J, U), \quad x \mapsto x_0 + \int_{t_0}^{t} f(s, x(s))\, ds,$$

for some smaller interval $J = [t_0 - \varepsilon, t_0 + \varepsilon] \subset I$. This is because, if x and f are continuous, so is $s \mapsto f(s, x(s))$, so the integral $\int_{t_0}^{t} f(s, x(s))\, ds$ is continuous (even C^1) and bounded on bounded closed intervals. The idea then is that, if f is also Lipschitz, then T **contracts** points for small $|t - t_0| \leq \varepsilon$:

$$\begin{aligned} |Tx(t) - Ty(t)| &= \Big| \int_{t_0}^{t} \big(f(s, x(s)) - f(s, y(s))\big)\, ds \Big| \leq \int_{t_0}^{t} \big|f(s, x(s)) - f(s, y(s))\big|\, ds \\ &\leq \int_{t_0}^{t} L|x(s) - y(s)|\, ds \leq L|t - t_0| \sup_{s \in J} |x(s) - y(s)| \end{aligned}$$

Thus, if $\varepsilon L < 1$, taking the maximum over $t \in J$ yields

$$\sup_{t \in J} |Tx(t) - Ty(t)| \leq \lambda \sup_{s \in J} |x(s) - y(s)|, \qquad \text{for } \lambda = \varepsilon L < 1,$$

so that Tx and Ty are closer to each other than x and y. As we shall now see, that gives us a local and unique solution of our problem.

[1] This is the reason why Lipschitz continuity is sometimes denoted by C^{1-}; Lipschitz is just slightly worse than being continuously differentiable. In this notation $Lip(I \times U, \mathbb{R}^n) = C^{0,1-}(I \times U, \mathbb{R}^n)$, to clarify that f is continuous with respect to its first variable and Lipschitz with respect to its second.

Well-posedness for the initial-value problem (IVP)

℘ The Picard - Lindelöf theorem

Let $f : I \times U \to \mathbb{R}^n$ be locally Lipschitz continuous with respect to its second variable and (t_0, x_0) a point in $I \times U$ determining the initial data. Then, for each $\eta > 0$, there exists $\varepsilon > 0$ such that the initial-value problem (IVP) has a unique solution $x \in C^1(\overline{B_\varepsilon(t_0)}, \overline{B_\eta(x_0)})$.

Proof (as outlined above)

Equivalence of formulations: If $x \in C^1(I, U)$ solves (IVP), then

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds,$$

by integration. Contrariwise, if $x \in C(I, U)$ fulfils the integral equation above, then x is a $C^1(I, U)$-solution (this follows from the Fundamental Theorem of Calculus). Thus, the initial-value problem (IVP) for $x \in C^1(I, U)$ is equivalent to (1) for $x \in C^0(I, U)$.

Some constants: Let $\delta > 0$ be such that $[t_0 - \delta, t_0 + \delta] \subset I$. Fix an arbitrary constant $\eta > 0$. Let

$$R := [t_0 - \delta, t_0 + \delta] \times \overline{B_\eta(x_0)}, \qquad M := \max_{(t,x)\in R} |f(t, x)|,$$

$$\varepsilon := \min\left\{\delta,\, \frac{\eta}{M},\, \frac{1}{2L}\right\}, \qquad J := [t_0 - \varepsilon, t_0 + \varepsilon],$$

where L denotes the Lipschitz constant for T in R (since R is compact, f is uniformly Lipschitz continuous on R).

Definition of T: For $v \in BC(J, \mathbb{R}^n)$, define

$$T(v)(t) := x_0 + \int_{t_0}^{t} f(s, v(s))\, ds, \qquad t \in J,$$

and consider

$$X := \{v \in BC(J, \mathbb{R}^n) \colon v(t_0) = x_0,\, \sup_{t\in J} |x_0 - v(t)| \le \eta\},$$

which is a closed subset of $BC(J, \mathbb{R}^n)$. Hence X is a complete metric space with the induced metric $d_\infty(v_1, v_2) = \sup_{t\in J} |v_1(t) - v_2(t)|$.

T maps X into X: If $v \in X$, then $T(v)(t_0) = x_0$ and

$$|x_0 - T(v)(t)| = \left| \int_{t_0}^{t} f(s, v(s))\, ds \right| \qquad \le |t - t_0| \max_{t\in J} |f(t, v(t))| \overset{v(t)\in\overline{B_\eta(x_0)}}{\le} \varepsilon M \le \eta,$$

by the definitions of R, M, ε and J.

T is a contraction on X: Let $v_1, v_2 \in X$. Then

$$\begin{aligned} |T(v_1)(t) - T(v_2)(t)| &= \left| \int_{t_0}^{t} \big(f(s, v_1(s)) - f(s, v_2(s))\big)\, ds \right| \\ &\le \varepsilon \max_{|s-t_0|\le|t-t_0|} \big|f(s, v_1(s)) - f(s, v_2(s))\big| \le \varepsilon L \max_{|s-t_0|\le|t-t_0|} |v_1(s) - v_2(s)| \\ &\le \tfrac{1}{2} \max_{s\in J} |v_1(s) - v_2(s)|, \end{aligned}$$

by the definition of ε. Now, taking the maximum over all $t \in J$ yields

$$d(T(v_1), T(v_2)) \leq \frac{1}{2} d(v_1, v_2).$$

Thus, according to the Banach fixed-point theorem, there exists a unique solution $x \in BC(J, \overline{B_\eta(x_0)})$.

℘ **Picard iteration**
Under the assumptions of the Picard-Lindelöf theorem, the sequence given by

$$x_0 = x(t_0), \qquad x_n = Tx_{n-1}, \quad n \in \mathbb{N}; \qquad (Tx)(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds,$$

converges uniformly and exponentially fast to the unique solution x on $J = [t_0 - \varepsilon, t_0 + \varepsilon]$:

$$\sup_J |x_n - x| \leq \frac{\lambda^n}{1-\lambda} \sup_J |x_1 - x_0|,$$

where $\lambda = \varepsilon L$ is the contraction constant used in the proof of the Picard-Lindelöf theorem.[1]

Proof
According to the proof of the Banach fixed-point theorem, if $m \geq n$ one has

$$d(x_n, x_m) \leq \frac{\lambda^n}{1-\lambda} d(x_1, x_0),$$

where $\lambda \in (0, 1)$ is the contraction constant. We apply this to the operator T, the metric d, and the constants ε and L as defined in the proof of the Picard-Lindelöf theorem. Since $\lim_{m\to\infty} x_m = x$ and $d(x_n, \cdot) = \|x_n - \cdot\|_{BC(J,\mathbb{R}^n)}$ is continuous, the proposition follows.

Ex.
The first Picard iteration for the initial-value problem

$$\dot{x} = \sqrt{x} + x^3, \qquad x(1) = 2,$$

is given by

$$x_1(t) = 2 + \int_1^t \left(\sqrt{2} + 2^3\right) ds = 2 + (\sqrt{2} + 8)(t - 1).$$

The second is

$$x_2(t) = 2 + \int_1^t \left(\sqrt{x_1(s)} + (x_1(s))^3\right) ds.$$

(This indicates that Picard iteration, in spite of its simplicity and fast convergence, is better suited as a theoretical and computer-aided tool, than as a way to solve ODE's by hand.)

2.8 Completions

Every metric space (and every normed space) can be made complete. To make this precise, recall that an **isomorphism** is a bijective (on-to-one and onto) map that preserves the essential structure of something. Likewise, an **isometry** is a map that preserves distances.[2]

[1] It is possible to refine both the estimate for λ and the size of the interval in different ways, but we will not pursue this here.

[2] The word *isos* is Greek for 'same', 'similar'; *morphe* is 'shape','form'; and *metron* is 'measure'.

Isometries

Two metric spaces (X, d_X) and (Y, d_Y) are called **isometric** if there exists a bijective isometry between them, i.e., if there exists an invertible function $\varphi\colon X \to Y$ such that

$$d_X(x_1, x_2) = d_Y(\varphi(x_1), \varphi(x_2)).$$

The function φ is called an **isometry**.

Ex.

- The set of sequences with only zeros and ones,

  $$X = \{(x_1, x_2, \ldots) \in l_\infty\colon \text{ for each } j,\ x_j = 0 \text{ or } x_j = 1\},$$

  endowed with the l_∞-metric,

  $$d(x, y) = \sup_{j\in\mathbb{N}} |x_j - y_j|$$

  is isometric to X endowed with the discrete metric, because

  $$d(x, y) = 1 \quad \text{unless} \quad x = y.$$

  The isometry is the identity operator, $\varphi\colon x \mapsto x, (X, \|\cdot\|_{l_\infty}) \to (X, d_{\text{discrete}})$.
- Let $a < b$. The metric spaces $(BC((0,1),\mathbb{R}), d_{(0,1)})$ and $(BC((a,b),\mathbb{R}), d_{(a,b)})$ of bounded continuous functions on the intervals $(0,1)$ and (a,b), respectively, endowed with the supremum metrics are isometric, since

  $$\varphi\colon BC((0,1),\mathbb{R}) \to BC((a,b),\mathbb{R}), \qquad f(\cdot) \overset{\varphi}{\mapsto} f\left(\frac{\cdot - a}{b-a}\right)$$

  is an isometry:

$$d_{(0,1)}(f,g) = \sup_{x\in(0,1)} |f(x) - g(x)| = \sup_{x\in(a,b)} \left| f\left(\frac{x-a}{b-a}\right) - g\left(\frac{x-a}{b-a}\right) \right| = d_{(a,b)}(\varphi(f), \varphi(g)).$$

  (Note that φ is invertible with inverse $\varphi^{-1}\colon f(\cdot) \mapsto f(a + \cdot(b-a))$. Hence, studying the metric space $BC((a,b),\mathbb{R})$ is no different from studying $BC((0,1),\mathbb{R})$.

Dense sets

A subset $M \subset X$ of a metric space is **dense** if its closure is the whole space.

$$M \text{ dense in } X \quad \overset{\text{def}}{\iff} \quad \overline{M} = X.$$

In this sense, M is 'almost all of X'.

Ex.

- $\mathbb{Q}$ is dense in $\mathbb{R}$:

  $$\text{for any } \lambda \in \mathbb{R} \text{ there is a sequence } \{q_n\}_n \subset \mathbb{Q} \quad \text{such that} \quad q_n \to \lambda \quad \text{in } \mathbb{R}.$$

- **Stone-Weierstrass**[1]: Let $I = [a, b]$ be a finite and closed interval. The polynomials, $P(\mathbb{R})$, the **infinitely continuously differentiable functions**, $C^\infty(I, \mathbb{R})$, and the k times continuously differentiable functions, $C^k(I, \mathbb{R})$, $k \geq 1$, are all dense in the space of bounded and continuous functions $BC(I, \mathbb{R})$:

  $$\forall\, \varepsilon > 0,\ f \in BC(I,\mathbb{R}) \quad \exists\, f_{\text{approx}} \in P(\mathbb{R}) \subset C^\infty(I,\mathbb{R}); \quad \sup_{x\in I} |f(x) - f_{\text{approx}}(x)| < \varepsilon.$$

[1] There are many versions of this theorem. The classical result states that polynomials are dense in $BC([a,b],\mathbb{C})$.

Separability

A metric space is said to be **separable** if it contains a countable dense set:

$$X \text{ separable} \quad \overset{\text{def}}{\iff} \quad \exists \{x_n\}_{n\in\mathbb{N}} \subset X; \qquad \overline{\{x_n\}_n} = X.$$

Ex.

- Since $\mathbb{Q}$ is countable, and $\overline{\mathbb{Q}} = \mathbb{R}$ (with respect to the distance $d(x,y) = |x-y|$), it follows that $(\mathbb{R}, |\cdot|)$ is separable.
- Using that $\overline{\mathbb{Q}} = \mathbb{R}$ one can show that all the spaces $\mathbb{R}^n, \mathbb{C}^n, l_p(\mathbb{R})$ and $l_p(\mathbb{C})$ for $1 \leq p < \infty, BC([a,b],\mathbb{R})$, and $BC([a,b],\mathbb{C})$ are separable (with respect to their standard norms/metrics).
- Neither l_∞ nor $BC((a,b),\mathbb{R})$ or $BC((a,b),\mathbb{C})$ is separable. (In this respect, these spaces are much 'bigger' than the other spaces considered in this course.)

Completion theorem

Every metric space is densely embedded in a complete metric space.

N.b. We do not prove this result but give some examples below.

Ex.

- If we complete $\mathbb{Q}$ with respect to the metric $d(x,y) = |x-y|$ we get $\mathbb{R}$:

$$\mathbb{Q} \overset{\text{dense}}{\hookrightarrow} \mathbb{R}.$$

- If we complete $C^\infty([0,1],\mathbb{R})$ with respect to the metric $d_\infty(f,g) = \sup_{[0,1]} |f-g|$, we get $BC([0,1],\mathbb{R})$.
- Let $I \subset \mathbb{R}$ be an open interval, and consider (measurable) functions such that the integral

$$\int_I |f(x)|^2 \, dx \quad \text{exists and is finite.}$$

Then

$$d_2(f,g) := \Big(\int_I |f(x) - g(x)|^2 \, dx \Big)^{1/2}$$

defines a metric on the set of these functions, so we have a metric space[1]. This space is complete and is called **the space of square-integrable functions**, written

$$L_2(I,\mathbb{R}).$$

The same space can be obtained by completing $BC(I,\mathbb{R})$ with respect to the d_2-metric. Hence

$$(BC(I,\mathbb{R}), d_2) \overset{\text{dense}}{\hookrightarrow} L_2(I,\mathbb{R}).$$

[1] Two functions in this space are equal if they are equal almost everywhere on I.

Chapter 3

Vector spaces and normed spaces

3.1 Vector spaces

Definition

A **real vector space** is a set X endowed with an operation called **addition**,

$$X \times X \to X, \qquad (x, y) \mapsto x + y,$$

an operation called **scalar multiplication**,

$$\mathbb{R} \times X \to X, \qquad (\lambda, x) \mapsto \lambda x,$$

an element $\mathbf{0} \in X$ called the **zero vector**, and for each $x \in X$ there exists an **additive inverse** $-x \in X$, such that for any elements $x, y, z \in X$ and real numbers $\lambda, \mu \in \mathbb{R}$ the following properties hold:

(i)	$x + \mathbf{0} = x,$	(additive identity)
(ii)	$x + (-x) = \mathbf{0},$	(additive inverse)
(iii)	$x + y = y + x,$	(symmetry)
(iv)	$x + (y + z) = (x + y) + z,$	(associativity)
(v)	$1x = x,$	(multiplicative identity)
(vi)	$\lambda(\mu x) = (\lambda\mu)x,$	(compatibility)
(vii)	$\lambda(x + y) = \lambda x + \lambda y,$	(distributivity)
(viii)	$(\lambda + \mu)x = \lambda x + \mu x,$	(distributivity)

The elements of X are called **vectors**. If the **field of scalars** $\mathbb{R}$ is replaced with $\mathbb{C}$ one obtains instead a **complex vector space**.[1]

N.b. 1 The notion of a vector space and that of a linear space are identical.

N.b. 2 The elements of a real vector space need not be real-valued. It is the *field of scalars* that determines whether a vector space is called real or complex.

[1] It is possible to define a vector space over any field $\mathbb{F}$, but we shall not use this.

Ex.

- $(\mathbb{R}, +, \cdot)$, the set of real numbers $\mathbb{R}$ endowed with the usual addition and multiplication is a real vector space.
- More generally, **Euclidean space**

$$\mathbb{R}^n = \{(x_1, \ldots, x_n) \colon x_j \in \mathbb{R} \text{ for } j = 1, 2, \ldots, n.\}$$

endowed with componentwise addition

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$$

and componentwise scalar multiplication

$$\lambda(x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n)$$

is a real vector space for any natural number $n \in \mathbb{N}$.

- The set of all bounded sequences l_∞ is a real vector space with componentwiseaddition and scalar multiplication. These operations transform bounded sequences into bounded sequences.
- **The set of real-valued continuous functions on an interval** $I \subset \mathbb{R}$,

$$C(I, \mathbb{R}) = \{f \colon I \to \mathbb{R} \text{ such that } f \text{ is continuous}\}$$

is a real vector space with the zero function $f \equiv 0$ as additive identity and $-f$ as additive inverse, when one defines

$$\begin{aligned} (f + g)(t) &:= f(t) + g(t), \\ (\lambda f)(t) &:= \lambda f(t), \\ (-f)(t) &:= -f(t). \end{aligned}$$

- All three examples above can be turned into complex vector spaces by replacing $\mathbb{R}$ with $\mathbb{C}$, i.e., when both the elements in the space and the field of scalars are replaced. These are the spaces

$$\mathbb{C}, \quad \mathbb{C}^n \quad \text{and} \quad C(I, \mathbb{C}).$$

The same spaces can also be considered as real vector spaces if the field of scalars is kept to be $\mathbb{R}$. Often, however, spaces that involve complex numbers are regarded as complex vector spaces.

- The essential property of a vector space is *linearity*: any line

$$\{(x, y) = (r\cos(\theta), r\sin(\theta)) \in \mathbb{R}^2 \colon \theta = \theta_0, r \in \mathbb{R}\}$$

is a vector space (addition and scalar multiplication as in $\mathbb{R}^2$), whereas a closed ball

$$\{(x, y) = (r\cos(\theta), r\sin(\theta)) \in \mathbb{R}^2 \colon 0 \leq r \leq \beta, 0 \leq \theta < 2\pi\}$$

is not (adding or scaling vectors might get one out of the space).

Linear subspaces

Let X be a vector space. A subset $S \subset X$ is a **subspace** of X if it is closed under linear operations, i.e.

$$S \subset X \text{ subspace of } X \quad \overset{\text{def}}{\iff} \quad \lambda x + \mu y \in S \quad \text{whenever} \quad x, y \in S \text{ and } \mu, \lambda \in \mathbb{R} \quad (\text{or } \mathbb{C}).$$

In particular, $\mathbf{0} \in S$, and S is itself a vector space (the axioms for a vector space follow from those of X).

Ex.

- In any vector space, $\{\mathbf{0}\}$ (the set consisting only of the zero element) is a subspace, since

$$\lambda\mathbf{0} + \mu\mathbf{0} = \mathbf{0} \in \{\mathbf{0}\} \quad \text{for all scalars } \lambda, \mu.$$

- Consider

$$\mathbb{R} = \{(x, 0, 0) \colon x \in \mathbb{R}\}$$

as a subset of

$$\mathbb{R}^3 = \{(x, y, z) \colon x, y, z \in \mathbb{R}\}.$$

Then $\mathbb{R}$ is a subspace of $\mathbb{R}^3$, since it is non-empty and

$$\lambda(x_1, 0, 0) + \mu(x_2, 0, 0) = (\lambda x_1 + \mu x_2, 0, 0) \in \mathbb{R} \subset \mathbb{R}^3.$$

- Similarly, the set of real-valued continuous functions on $\mathbb{R}$ which vanish on some set $S \subset \mathbb{R}$ is a subspace of $C(\mathbb{R}, \mathbb{R})$:

$$\{f \in C(\mathbb{R}, \mathbb{R}) \colon f \equiv 0 \text{ on } S\} \quad \text{is a subspace of} \quad C(\mathbb{R}, \mathbb{R}),$$

since

$$\mu f(x) + \lambda g(x) = 0 \quad \text{if} \quad f(x) = 0 \text{ and } g(x) = 0.$$

- The vector space of **polynomials of degree at most** n, $P_n(\mathbb{R})$, endowed with the usual addition and scalar multiplication, is a subspace of the set of polynomials of degree at most $n + 1$. Indeed,

$$P_0(\mathbb{R}) \subset P_1(\mathbb{R}) \subset \ldots \subset P_n(\mathbb{R}) \subset P_{n+1}(\mathbb{R}) \subset \ldots \subset P(\mathbb{R}) := \bigcup_{n=0}^{\infty} P_n(\mathbb{R})$$

are all subspaces of each other and, ultimately, of **the vector space of all polynomials**, $P(\mathbb{R})$.

Span

Let X be a vector space, and $S \subset X$ any subset of X. A **linear combination** of vectors $u_1, \ldots, u_n$ is a finite sum

$$\sum_{j=1}^{n} a_j u_j,$$

where $a_1, \ldots, a_n$ are scalars. The **(linear) span** of $S \subset X$ is the set of all linear combinations of vectors in S:

$$\operatorname{span}(S) \overset{\text{def.}}{=} \Big\{ \sum_{\text{finite}} a_j x_j \colon x_j \in S, a_j \text{ scalars} \Big\}.$$

For convenience, we define $\operatorname{span}(\emptyset) \overset{\text{def.}}{=} \{\mathbf{0}\}$. If $V = \operatorname{span}(S)$ we say that S **generates** V.

℘ **The linear span of a set S is the smallest subspace containing S**

For any $S \subset X$, $\operatorname{span}(S)$ is a subspace of X, and

$$\operatorname{span}(S) = \bigcap_{S \subset V} \{V \colon V \text{ is a subspace of } X\}.$$

Ex.

- Let $x = (1,0)$, $y = (2,0)$ and $z = (1,1)$ be vectors in $\mathbb{R}^2$. Then

$$\operatorname{span}\{x\} = \operatorname{span}\{y\} = \operatorname{span}\{x,y\} = \{(\lambda,0)\colon \lambda \in \mathbb{R}\},$$

$$\operatorname{span}\{z\} = \{(\lambda,\lambda)\colon \lambda \in \mathbb{R}\},$$

$$\operatorname{span}\{x,z\} = \operatorname{span}\{y,z\} = \operatorname{span}\{x,y,z\} = \mathbb{R}^2.$$

- The vectors $e_1 = (1,0,\ldots)$, $e_2 = (0,1,0,\ldots)$, …, $e_n = (0,\ldots,0,1)$ generate $\mathbb{R}^n$.
- In general, the span of a set differs between real and complex vector spaces:

$$\operatorname{span}_{\mathbb{R}}\{1\} = \mathbb{R} \quad \text{but} \quad \operatorname{span}_{\mathbb{C}}\{1\} = \mathbb{C}.$$

3.2 Bases and dimension

Let X be a vector space.

Linear dependence

A sequence of vectors $u_1, u_2, \ldots$ in X is called **linearly dependent** if one of them is linear combination of some of the others:

$$\{u_1,u_2,\ldots\} \text{ linearly dependent} \quad \overset{\text{def}}{\iff} \quad \sum_{j=1}^{n} a_j u_j = \mathbf{0} \quad \text{for some } n \in \mathbb{N} \text{ and at least one } a_j \neq 0.$$

Else, the sequence is **linearly independent**:

$$\{u_1,u_2,\ldots\} \text{ linearly independent} \quad \overset{\text{def}}{\iff} \quad \Big[\text{for any } n \in \mathbb{N}: \quad \sum_{j=1}^{n} a_j u_j = \mathbf{0} \implies a_j = 0 \,\forall\, j\Big].$$

More generally, a set S is linearly independent if all finite subsets of it are linearly independent.

Ex.

- The vectors $x = (1,0)$, $y = (2,0)$ and $z = (1,1)$ are linearly dependent in $\mathbb{R}^2$, since

$$\begin{bmatrix}2\\0\end{bmatrix} = 2\begin{bmatrix}1\\0\end{bmatrix}.$$

  But both the sets $\{x,z\}$ and $\{y,z\}$ are linearly independent, since

$$a_1 x + a_2 z = \mathbf{0} \iff a_1\begin{bmatrix}1\\0\end{bmatrix} + a_2\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \iff \begin{bmatrix}a_1+a_2\\a_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \iff a_1 = a_2 = 0,$$

  and similarly for $\{y,z\}$.
- If $\mathbf{0} \in S$, then S is linearly dependent.
- $\{1, x, x^2, \ldots\}$ is linearly independent in $P(\mathbb{R})$.
- $\{1, \cos(x), \sin(x), \cos(2x), \sin(2x), \ldots\}$ is linearly independent in $C(I,\mathbb{R})$.

Hamel Bases

A linearly independent set which generates X is called a **(Hamel) basis** for X:

$$S \subset X \text{ Hamel basis for } X \quad \overset{\text{def}}{\iff} \quad \operatorname{span}(S) = X \quad \text{and} \quad S \text{ lin. indep.}$$

Equivalently, S is a Hamel basis for X if every vector $x \in X$ has a *unique and finite* representation

$$x = \sum_{\text{finite}} a_j u_j, \qquad u_j \in S.$$

We shall consider only **ordered** Hamel bases, in which case the scalars a_j, called **coordinates**, are well defined.

Ex.

- $\{e_1, \ldots, e_n\}$, with

$$e_j = (0, \ldots, \underbrace{1}_{\text{jth position}}, 0 \ldots)$$

  is called the **standard basis** for $\mathbb{R}^n$.
- $\{1, x, x^2, \ldots\}$ is an ordered Hamel basis for $P(\mathbb{R})$: every real polynomial can be uniquely expressed as a finite sum,

$$p(x) = \sum_{\text{finite}} a_j x^j, \qquad a_j \in \mathbb{R}.$$

Dimension

If X has a basis consisting of finitely many vectors, X is said to be **finite-dimensional**. Else, X is **infinite-dimensional**.

℘ **The dimension of any finite-dimensional vector space is well-defined**

All bases of a finite-dimensional vector space have the same number of elements. This number is called the **dimension** of the space.

Proof

Suppose that $\{e_j\}_{j=1}^m$ and $\{f_j\}_{j=1}^n$ are both bases, and that $m > n$. Since a basis is linearly independent, the only solution of

$$\sum_{j=1}^{m} a_j e_j = \mathbf{0} \quad \text{ought to be} \quad a_1, \ldots, a_m = 0.$$

Since $\{f_j\}_{j=1}^n$ is also a basis, we may represent $e_j = \sum_{k=1}^n b_{j,k} f_k$ in a unique way. Then

$$\sum_{j=1}^{m}\sum_{k=1}^{n} a_j b_{j,k} f_k = 0 \quad \text{meaning that} \quad \sum_{j=1}^{m} a_j b_{j,k} = 0 \text{ for } k = 1, \ldots, n.$$

This is a linear homogeneous system with n equations and $m > n$ unknowns (the scalars a_j). Such a system always has a non-trivial solution (meaning that some $a_j \neq 0$). Hence $\{e_j\}_{j=1}^m$ is not linearly independent, so it cannot be a basis.

Ex.

- $\mathbb{R}^n$ has dimension n.
- $P_n(\mathbb{R})$, has dimension $n+1$ since $\{1, x, x^2, ..., x^n\}$ is a basis for $P_n(\mathbb{R})$.
- $\mathbb{C}^n$ has dimension n when considered as a *complex* vector space, but $2n$ when considered a *real* vector space.
- The l_∞, BC, and $P(\mathbb{R})$ are all infinite-dimensional spaces.

Vector space isomorphisms

A **vector space isomorphism** is a bijective linear map between two vector spaces, i.e. an invertible function $T\colon X \to Y$ such that

$$T(x+y) = Tx + Ty \quad \text{and} \quad T(\lambda x) = \lambda Tx \qquad \text{for all} \quad x, y \in X,\ \lambda \in \mathbb{R}\ (\text{or } \mathbb{C}).$$

Two vector spaces which allow for such a mapping are called **isomorphic**, and we write

$$X \cong Y \quad \overset{\text{def}}{\iff} \quad \exists \text{ isomorphism } T\colon X \to Y.$$

N.b. A set which is the image of a vector space isomorphism automatically becomes a vector space (it inherits its linear structure from X).

Ex.

- Regarded as a real vector space, the space $\mathbb{C}^n$ of complex n-tuples,

  $$z = (z_1, \dots, z_n), \qquad z_1, \dots, z_n \in \mathbb{C}$$

  is isomorphic to Euclidean space $\mathbb{R}^{2n}$ via the isomorphism[1]

  $$z = (x_1 + iy_1, \dots, x_n + iy_n) \mapsto (x, y) = (x_1, \dots, x_n, y_1, \dots, y_n).$$

- The set of polynomials with real coefficients of degree at most n, $P_n(\mathbb{R})$, is isomorphic to $\mathbb{R}^{n+1}$. The mapping

  $$T\colon P_n(\mathbb{R}) \to \mathbb{R}^{n+1}, \qquad a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0 \mapsto (a_0, a_1, \dots, a_n)$$

  is both *bijective*,

  $$\text{for any } (a_0, \dots, a_n) \in \mathbb{R}^{n+1} \text{ there exists a unique } p(x) = \sum_{k=0}^{n} a_k x^k \in P_n(\mathbb{R}),$$

  and *linear*,

  $$\begin{aligned} T\Big(\lambda \sum_{k=0}^{n} a_k x^k + \mu \sum_{k=0}^{n} b_k x^k\Big) &= T\Big(\sum_{k=0}^{n} (\lambda a_k + \mu b_k) x^k\Big) \\ &= (\lambda a_0 + \mu b_0, \dots, \lambda a_n + \mu b_n) = \lambda(a_0, \dots, a_n) + \mu(b_0, \dots, b_n) \\ &= \lambda T\Big(\sum_{k=0}^{n} a_k x^k\Big) + \mu T\Big(\sum_{k=0}^{n} b_k x^k\Big). \end{aligned}$$

  Hence any linear operation in $P_n(\mathbb{R})$ corresponds to a linear operation in $\mathbb{R}^{n+1}$, meaning that $P_n(\mathbb{R})$ and $\mathbb{R}^{n+1}$ are isomorphic as vector spaces.

$\wp$ Any finite-dimensional vector space is isomorphic to Euclidean space

Let X be a real vector space with basis $\{e_1, \dots, e_n\}$. Then $X \cong \mathbb{R}^n$.

Proof

By the definition of a basis, any $x \in X$ has a unique representation $x = \sum_{j=1}^{n} a_j e_j$. Let $T\colon X \to \mathbb{R}^n$ be the mapping defined by

$$Tx = (a_1, \dots, a_n).$$

[1] This is also an isometry, since $|z|^2 = |x|^2 + |y|^2$.

T **is linear:** if $x = \sum a_j e_j$ and $y = \sum b_j e_j$,

$$T(\lambda x + \mu y) = (\lambda a_1 + \mu b_1, \ldots, \lambda a_n + \mu b_n) = \lambda(a_1, \ldots, a_n) + \mu(b_1, \ldots, b_n) = \lambda Tx + \mu Ty,$$

T **is surjective:**

$$\text{for any } (a_1, \ldots, a_n) \in \mathbb{R}^n \quad \text{there exists} \quad x = \sum_{j=1}^{n} a_j e_j;\ Tx = (a_1, \ldots, a_n)$$

T **is injective:**

$$Tx = Ty \iff \forall j\colon a_j = b_j \implies x = y.$$

Thus T is a vector space isomorphism.

N.b. In an n-dimensional vector space, $m > n$ vectors are linearly dependent.

3.3 Normed spaces

Definition

A **normed space** is a vector space X endowed with a function

$$X \to [0, \infty), \qquad x \mapsto \|x\|,$$

called the **norm** on X, which satisfies:

(i)	$\|\lambda x\| = \lvert\lambda\rvert \, \|x\|$,	(positive homogeneity)
(ii)	$\|x + y\| \le \|x\| + \|y\|$,	(triangle inequality)
(iii)	$\|x\| = 0$ if and only if $x = 0$,	(positive definiteness)

for all scalars λ and all elements $x, y \in X$. A vector space may allow for many different norms.

Ex. The vector space $\mathbb{R}^n$ with the usual addition and scalar multiplication allows for several norms, for example: the **Euclidean norm**

$$\|(x_1, \ldots, x_n)\|_2 = \left(x_1^2 + \ldots + x_n^2\right)^{1/2}$$

the **maximum norm**

$$\|(x_1, \ldots, x_n)\|_\infty = \max\{|x_1|, \ldots, |x_n|\},$$

and the **summation norm**

$$\|(x_1, \ldots, x_n)\|_1 = |x_1| + \ldots + |x_n|.$$

These are all special cases of the (finite-dimensional) l_p**-norm**

$$\|(x_1, \ldots, x_n)\|_p = \Big(\sum_{j=1}^{n} |x_j|^p\Big)^{1/p}, \quad 1 \le p \le \infty.$$

More generally, sequence l_p**-space** is defined as sequence of real (complex) numbers for which

$$\|x\|_p = \Big(\sum_{j=1}^{\infty} |x_j|^p\Big)^{1/p} < +\infty, \quad 1 \le p \le \infty.$$

It is a normed vector space.

Proof (of the example)
For all l_p-norms it is clear that they are non-negative functions, and that

$$\|x\| = 0 \quad \Longleftrightarrow \quad x = (x_1, \ldots, x_n) = (0, \ldots, 0).$$

In addition,

$$\|\lambda x\|_p = \big((\lambda x_1)^p + \cdots + (\lambda x_n)^p\big)^{1/p} = |\lambda|\big(x_1^p + \cdots + x_n^p\big)^{1/p} = |\lambda|\|x\|_p,$$

and similarly for $\|\cdot\|_\infty$.

The **triangle inequality** for $\|\cdot\|_{l_\infty}$ and $\|\cdot\|_{l_1}$ follows from that on $\mathbb{R}$:

$$\|x+y\|_{l_1} = \sum_{j=1}^n |x_j + y_j| \leq \sum_{j=1}^n (|x_j| + |y_j|) = \sum_{j=1}^n |x_j| + \sum_{j=1}^n |y_j| = \|x\|_{l_1} + \|y\|_{l_1},$$

$$\begin{aligned} \|x+y\|_{l_\infty} &= \max\{|x_1 + y_1|, \ldots, |x_n + y_n|\} \\ &\leq \max\{|x_1| + |y_1|, \ldots, |x_n| + |y_n|\} \\ &\leq \max\{|x_1|, \ldots, |x_n|\} + \max\{|y_1|, \ldots, |y_n|\} \\ &= \|x\|_{l_\infty} + \|y\|_{l_\infty}. \end{aligned}$$

The triangle inequality for $\|\cdot\|_{l_2}$ is a consequence of the **Cauchy-Schwarz inequality**, which we will prove later.

For general $p > 1$ the triangle inequality is known as the **Minkowski inequality**

$$\left(\sum_{j=1}^n |x_j + y_j|^p\right)^{1/p} \leq \left(\sum_{j=1}^n |x_j|^p\right)^{1/p} + \left(\sum_{j=1}^n |y_j|^p\right)^{1/p}.$$

Ex. The space of real- (or complex-) valued **bounded and continuous functions** on an interval (open or closed), $BC(I, \mathbb{R})$, becomes a normed vector space when endowed with the **supremum norm**

$$\|f\|_\infty = \sup_{x \in I} |f(x)|.$$

If $I = [a, b]$ it follows from the *extreme value theorem* that $BC([a,b], \mathbb{R}) = C([a,b], \mathbb{R})$ (as sets and linear spaces) and

$$\|f\|_\infty = \sup_{x \in [a,b]} |f(x)| = \max_{x \in [a,b]} |f(x)|.$$

If $I = (a, b)$ is either infinite or does not contain its end points, then $BC((a,b), \mathbb{R}) \subsetneq C((a,b), \mathbb{R})$. An example of this strict inclusion is the function $x \mapsto 1/x$ on (0,1). It is continuous, but

$$[x \mapsto 1/x] \notin BC((0,1), \mathbb{R}),$$

since $\sup_{x \in (0,1)} |1/x| = \infty$.

℘ **Normed spaces are metric spaces**
If $\|\cdot\|$ is a norm on X, then $d(x, y) := \|x - y\|$ is a metric on X.

Proof
The distance is non-negative and well defined, since

$$0 \leq \underbrace{\|x-y\|}_{d(x,y)} \leq \|x\| + \|y\| < \infty, \qquad \text{for} \qquad x, y \in (X, \|\cdot\|).$$

Symmetry:	$d(x,y) = \|x-y\| = \|y-x\| = d(y,x).$
Triangle inequality:	$d(x,y) = \|x-y\| \leq \|x-z\| + \|z-y\| = d(x,z) + d(z,y).$
Non-degeneracy:	$d(x,y) = \|x-y\| = 0 \iff x-y=0 \iff x=y.$

N.b. Metric spaces need not be vector spaces. The set of positive real numbers, $\mathbb{R}_+ = (0,\infty)$, with the metric given by $d(x,y) := |x-y|$ is a metric space, but it is not a linear space, since it contains neither an additive identity (0) nor additive inverses $(-x)$.

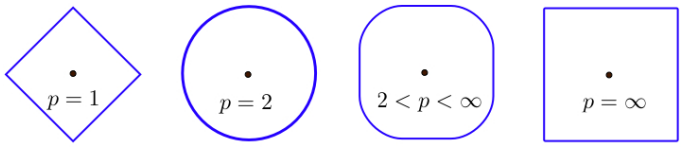
Unit balls and spheres

For normed spaces, or other vector spaces that are also metric spaces,

$$B_r := B_r(0) \qquad \text{and} \qquad S_r = S_r(0),$$

denote balls and spheres centered at the origin (zero element). The sets B_1 and S_1 are called the **unit ball** and **unit sphere**, respectively.

N.b. The unit ball may look quite different depending on the underlying metric/norm. The following illustration captures this in the case of the l_p-norm on $\mathbb{R}^2$. Homogeneity and the triangle inequality however imply that a ball in any metric given by a norm will always be a convex set in the underlying space.



The unit sphere for different metrics: $\|x\|_{l_p} = 1$ in $\mathbb{R}^2$.

Equivalence of norms

Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a vector space X are said to be **equivalent** if there exists a number $c \in \mathbb{R}$ such that

$$c^{-1}\|x\|_1 \leq \|x\|_2 \leq c\|x\|_1 \qquad \text{for all} \qquad x \in X.$$

Ex. The maximum and summation norms are equivalent on $\mathbb{R}^n$, since

$$\max_{1\leq j\leq n} |x_j| \leq \sum_{j=1}^{n} |x_j| \qquad \text{and} \qquad \sum_{j=1}^{n} |x_j| \leq n \max_{1\leq j\leq n} |x_j|.$$

Hence

$$n^{-1}\|x\|_{l_\infty} \leq \|x\|_{l_1} \leq n\|x\|_{l_\infty} \qquad \text{for} \qquad x = (x_1, \ldots, x_n).$$

℘ **On a finite-dimensional vector space any two norms are equivalent**
Let X be a finite dimensional vector space and $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on X. Then there exists c such that $c^{-1}\|x\|_1 \leq \|x\|_2 \leq c\|x\|_1$.

In particular, any norm on $\mathbb{R}^n$ is equivalent to the Euclidean norm.

Proof

Let $\{e_1, ..., e_n\}$ be a basis for X, we define a new norm on X by

$$\|\sum_{j=1}^{n} a_j e_j\| = \left(\sum_{j=1}^{n} |a_j|^2\right)^{1/2}.$$

It is enough to show that both norms are equivalent to that one.

Let S_1 be the unit sphere of X with the new norm $\|\cdot\|$. Then $x \in S_1$ if and only if

$$x = \sum a_j e_j \quad \text{where} \quad \sum_{j=1}^{n} a_j^2 = 1.$$

Then S_1 is a compact subset of the metric space $(X, \|\cdot\|)$.

Consider

$$\|x\|_1 = \left\|\sum_{j=1}^{n} a_j e_j\right\|_1 \leq \sum_{j=1}^{n} |a_j| \|e_j\|_1 \leq \sum_j \|e_j\|_1 =: C_1, \quad x \in S_1,$$

since $|a_j| \leq 1$. Then by the positive homogeneity of the norms

$$\|x\|_1 \leq C_1 \|x\|, \quad \text{and} \quad \|x - y\|_1 \leq C_1 \|x - y\|.$$

Thus $\|\cdot\|_1$ is a (Lipschitz) continuous function on X, it is bounded and attains its minimum on S_1. Non-degeneracy of the norm implies that $c^{-1} \leq \|x\|_1 \leq c$ for $x \in S_1$. Hence $c^{-1}\|x\| \leq \|x\|_1 \leq c\|x\|$. Thus $\|\cdot\|$ and $\|\cdot\|_1$ are equivalent. Similarly $\|\cdot\|$ and $\|\cdot\|_2$ are equivalent and then $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.

Ex.

- If $\|\cdot\|_1$ and $\|\cdot\|_2$ are two equivalent norms in a vector space X then a sequence x_k converges to x_0 in X in the metric defined by the first norm (i.e., $\|x_k - x_0\|_1 \to 0,\ k \to \infty$) if and only if it converges in the metric defined by the second norm, $\|x_k - x_0\|_2 \to 0,\ k \to \infty$. In finite dimensional vector space the convergence does not depend on the choice of a norm.
- The situation is different in infinite-dimensional space. For example let l_0 be the vector space of the sequences that have only finite number non-zero terms. This set can be endowed with different norms $\|\cdot\|_p,\ 1 \leq p \leq \infty$. Consider $x_n = (\underbrace{1/n, ..., 1/n}_{n times}, 0, 0, ...)$.

  Then $x_n \to 0$ in $(l_0, \|\cdot\|_\infty)$ since $\|x_n\|_\infty = 1/n$ but $x_n \not\to 0$ in $(l_0, \|\cdot\|_1)$ since $\|x_n\_1 = 1$.

In finite-dimensional space the unit sphere is compact. It is no longer true in infinite dimensional spaces. For any $p \geq 1$, let l_p be the space of sequences $\{x_j\}_{j\in\mathbb{N}} = (x_1, x_2, \ldots)$ for which

$$\|x\|_p = \big(\sum_{j\in\mathbb{N}} |x_j|^p\big)^{1/p} < \infty.$$

Let further

$$e_1 = (1, 0, 0, \ldots), \quad e_2 = (0, 1, 0, 0 \ldots) \quad \text{and} \quad e_j = (\ldots, 0, 1, 0, \ldots), \quad j \in \mathbb{N}.$$

Then

$$e_j \in S_1 \quad \text{for all} \quad j \in \mathbb{N},$$

but

$$d(e_i, e_j) = \|e_i - e_j\|_p = (|1|^p + |-1|^p)^{1/p} = 2^{1/p} \geq 1 \quad \text{whenever} \quad i \neq j.$$

3.4 Banach spaces and Schauder bases

Banach spaces

A normed space is called a **Banach space** if it is a complete metric space (with the metric induced by the norm).

Ex.

- $\mathbb{R}^n$ is a Banach space with any l_p-norm, $1 \leq p \leq \infty$.
- l_p-spaces are Banach spaces, $1 \leq p \leq \infty$.
- $BC(I, \mathbb{R})$ is a Banach space.
- Polynomials $P(\mathbb{R})$, considered as continuous bounded functions on $[0, 1]$, form a subspace of $BC([0, 1], \mathbb{R})$, this subspace is not closed and hence not complete.

Whereas the concept of a Hamel basis is very general - it applies to any vector space - it is not particularly well suited for infinite-dimensional Banach spaces.

Infinite-dimensional Banach spaces have only uncountable Hamel bases

Fact Let X be an infinite-dimensional Banach space. Then a sequence $\{e_j\}_{j\in\mathbb{N}}$ cannot be a Hamel basis for X.

Schauder Bases

Let $(X, \|\cdot\|)$ be a Banach space. A sequence $\{e_j\}_{j\in\mathbb{N}}$ is called a **Schauder basis** (or **countable basis**) for X if every vector $x \in X$ has a *unique* representation

$$x = \sum_{j\in\mathbb{N}} x_j e_j,$$

meaning that $\lim_{N\to\infty} \|x - \sum_{j=1}^{N} x_j e_j\| = 0$. The scalars x_j are the **coordinates** of x.

N.b. Any space with a Schauder basis is separable.[1]

*From now on, the word **basis** refers to an ordered basis, Hamel (in the case of any finite-dimensional vector space) or Schauder (in the case of any infinite-dimensional Banach space).*

In both cases, a basis assigns to each $x \in X$ unique coordinates $x_1, x_2, \ldots$.

Ex.

- The vectors $(1, 0, 0), (1, 1, 0), (1, 1, 1)$ provide a (Hamel) basis for $\mathbb{R}^3$. Find the coordinates $[c_1, c_2, c_3]$ of $(2, 0, 1)$ in this basis.

$$c_1 \begin{bmatrix}1\\0\\0\end{bmatrix} + c_2 \begin{bmatrix}1\\1\\0\end{bmatrix} + c_3 \begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}2\\0\\1\end{bmatrix} \iff \begin{bmatrix}c_1+ & c_2+ & c_3\\ & c_2+ & c_3\\ & & c_3\end{bmatrix} = \begin{bmatrix}2\\0\\1\end{bmatrix} \iff \begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}2\\-1\\1\end{bmatrix}.$$

  The coordinates for $(2, 0, 1)$ in the new basis are $[2, -1, 1]$.
- The trigonometric functions $\{e^{ikx}\}_{k\in\mathbb{Z}}$ is a (Schauder) basis for $L_2((-\pi, \pi), \mathbb{C})$. The coordinates in this basis are known as **Fourier coefficients**.

[1] The opposite is not true; there are (strange) separable Banach spaces with no Schauder basis.

Ex. Let $e_j = (0, \dots, \underbrace{1}_{\text{jth position}}, 0 \dots)$. Then $\{e_j\}_{j=1}^{\infty}$ is a (Schauder) basis for l_p, $1 \le p < \infty$:[1]

Approximation property:

$$x = \{x_j\}_{j\in\mathbb{N}} \in l_p \quad \Longrightarrow \quad \sum_{j=1}^{\infty} |x_j|^p < \infty \quad \Longrightarrow \quad \sum_{N+1}^{\infty} |x_j|^p \to 0 \text{ as } N \to \infty.$$

Thus

$$\Big\| \sum_{j=1}^{N} x_j e_j - x \Big\|_p = \|(x_1, \dots, x_N, 0, 0, \dots) - (x_1, \dots, x_N, x_{N+1}, \dots)\|_{l_p} = \Big(\sum_{N+1}^{\infty} |x_j|^p \Big)^{1/p}$$

Hence $\sum_{j=1}^{N} \to x$ in l^p as $N \to \infty$. **Uniqueness of coordinates:**

$$\sum_{j=1}^{\infty} x_j e_j = \sum_{j=1}^{\infty} y_j e_j \quad \Longleftrightarrow \quad \lim_{N\to\infty} \sum_{j=1}^{N} |x_j - y_j|^p = 0 \quad \Longrightarrow \quad x_j = y_j, \text{ for all } j \in \mathbb{N}.$$

Isomorphisms

Isomorphisms on normed spaces

If the vector spaces are normed, they are **isomorphic as normed spaces** if they are isomorphic (as vector spaces) and isometric (as metric spaces). Sometimes this is called **isometrically isomorphic** to avoid confusion.

Ex.

- The spaces $BC((0,1),\mathbb{R})$ and $BC((a,b),\mathbb{R})$ are isometrically isomorphic, since the isometry φ in the example above is also a vector space isomorphism:

$$\varphi(\lambda f + \mu g)(x) = (\lambda f + \mu g)\Big(\frac{x-a}{b-a}\Big) = \lambda f\Big(\frac{x-a}{b-a}\Big) + \mu g\Big(\frac{x-a}{b-a}\Big) = \lambda\varphi(f)(x) + \mu\varphi(g)(x).$$

Embeddings

If a (normed) vector space X is isomorphic to a subspace $M \subset Y$ of another (normed) vector space, we say that it is **(continuously) embedded** in Y,

$$X \hookrightarrow Y \quad \overset{\text{def}}{\Longleftrightarrow} \quad \exists \text{ isomorphism } T\colon X \to M \subset Y.$$

To ease terminology we shall use this concept also for isometries between metric spaces.

Ex.

- The vector space of polynomials of degree at most 1, $P_1(\mathbb{R})$, is continuously embedded in three-dimensional Euclidean space,

$$P_1 \hookrightarrow \mathbb{R}^3, \quad \text{since} \quad P_1 \cong \mathbb{R}^2 \subset \mathbb{R}^3.$$

- The identity operator provides an embedding of the vector space of continuously differentiable functions on an interval I into the vector space of continuous functions on the same interval:[2]

$$C^1(I,\mathbb{R}) \hookrightarrow C(I,\mathbb{R}).$$

[1] l_∞ has no Schauder basis since it is not separable

[2] Note that, when $I = \mathbb{R}$, or if I is not closed, neither of these spaces are normed. When $I = [a,b]$ is closed and finite, $BC([a,b],\mathbb{R}) = (C([a,b],\mathbb{R}), \|\cdot\|_\infty)$. Otherwise, $BC(I,\mathbb{R})$ is strictly smaller than $C(I,\mathbb{R})$.

Completion theorem for normed spaces

Fact Any normed space is densely embedded in a Banach space.

Ex. The following examples we had in the previous chapter provide embeddings of normed spaces (not only metric spaces)

$$\mathbb{Q} \overset{\text{dense}}{\hookrightarrow} \mathbb{R}.$$

$$(P([0,1],\mathbb{R}), \|\cdot\|_\infty) \overset{\text{dense}}{\hookrightarrow} BC([0,1],\mathbb{R}).$$

$$(BC(I,\mathbb{R}), \|\cdot\|_2) \overset{\text{dense}}{\hookrightarrow} L_2(I,\mathbb{R}).$$

3.5 Inner-product spaces

Let X be a vector space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. The complex conjugate of a complex number $z = x + iy$ is $\overline{z} = x - iy$, in particular, $\overline{x} = x$ for $x \in \mathbb{R}$.

Inner-product spaces

An **inner product** $\langle \cdot, \cdot \rangle$ on X is a map $X \times X \to \mathbb{K}$, $(x, y) \mapsto \langle x, y \rangle$, that is **conjugate symmetric**

$$\langle x, y \rangle = \overline{\langle y, x \rangle},$$

linear in its first argument,

$$\langle \lambda x, y \rangle = \lambda \langle x, y \rangle,$$
$$\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle,$$

and **non-degenerate (positive definite)**,

$$\langle x, x \rangle > 0 \quad \text{for} \quad x \neq 0,$$

with $x, y, z \in X$ and $\lambda \in \mathbb{K}$ arbitrary. The pair $(X, \langle \cdot, \cdot \rangle)$ is called an **inner-product space**.

Ex.

- The canonical inner product is the dot product in $\mathbb{R}^n$:

$$\langle x, y \rangle := x \cdot y = \sum_{j=1}^{n} x_j y_j.$$

- For matrices in $M_{n\times n}(\mathbb{R})$ one can define a dot product by setting

$$\langle A, B \rangle := \operatorname{tr}(B^t A),$$

where $\operatorname{tr}(C) = \sum_{j=1}^{n} c_{jj}$ is the trace of a matrix C, and B^t is the transponse of B. Then

$$B^t A = \sum_{j=1}^{n} b_{ij}^t a_{jk} = \sum_{j=1}^{n} b_{ji} a_{jk},$$

and

$$\operatorname{tr}(B^t A) = \sum_{k=1}^{n} \sum_{j=1}^{n} b_{jk} a_{jk} = \sum_{1 \leq j,k \leq n} a_{jk} b_{jk}$$

coincides with the dot product on $\mathbb{R}^{nn} \cong M_{n\times n}(\mathbb{R})$.

Properties of the inner product
An inner product satisfies

$$\begin{aligned}
&\text{(i)} && \langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle,\\
&\text{(ii)} && \langle x, \lambda y\rangle = \bar{\lambda}\langle x, y\rangle,\\
&\text{(iii)} && \langle x, 0\rangle = \langle 0, x\rangle = 0,\\
&\text{(iv)} && \text{If } \langle x, z\rangle = 0 \text{ for all } z \in X \quad \text{then} \quad x = 0.
\end{aligned}$$

N.b. By linearity, the last property implies that if $\langle x, z\rangle = \langle y, z\rangle$ for all $z \in X$, then $x = y$.

Proof
(i)

$$\langle x, y+z\rangle = \overline{\langle y+z, x\rangle} = \overline{\langle y, x\rangle + \langle z, x\rangle} = \overline{\langle y, x\rangle} + \overline{\langle z, x\rangle} = \langle x, y\rangle + \langle x, z\rangle.$$

(ii)

$$\langle x, \lambda y\rangle = \overline{\langle \lambda y, x\rangle} = \overline{\lambda\langle y, x\rangle} = \bar{\lambda}\langle x, y\rangle.$$

(iii)

$$\langle \mathbf{0}, x\rangle = \langle 0x, x\rangle = 0\langle x, x\rangle = 0,$$

and

$$\langle x, \mathbf{0}\rangle = \overline{\langle 0, x\rangle} = 0.$$

(iv)

$$\langle x, z\rangle = 0 \text{ for all } z \in X \quad \Longrightarrow \quad \langle x, x\rangle = 0 \quad \Longrightarrow \quad x = 0.$$

Inner-product spaces as normed spaces

An inner-product space $(X, \langle\cdot,\cdot\rangle)$ carries a natural norm given by $\|x\| := \langle x, x\rangle^{1/2}$. To prove this, we need:

℘ **The Cauchy - Schwarz inequality**
For all $x, y \in (X, \langle\cdot,\cdot\rangle)$,

$$|\langle x, y\rangle| \leq \|x\|\|y\|,$$

with equality if and only if x and y are linearly dependent.

Proof
Linearly dependent case: Without loss of generality, assume that $x = \lambda y$ (if $y = \lambda x$ we can always relabel the vectors). Then

$$\begin{aligned}
|\langle x, y\rangle| &= |\langle \lambda y, y\rangle| = |\lambda|\langle y, y\rangle\\
&= |\lambda|\|y\|^2 = \|\lambda y\|\|y\| = \|x\|\|y\|.
\end{aligned}$$

Linearly independent case: If $x - \lambda y \neq 0$ and $y - \lambda x \neq 0$ for all $\lambda \in \mathbb{K}$, then also $x, y \neq 0$, and

$$\begin{aligned}
0 &< \langle x + \lambda y, x + \lambda y\rangle\\
&= \langle x, x + \lambda y\rangle + \lambda\langle y, x + \lambda y\rangle\\
&= \langle x, x\rangle + \langle x, \lambda y\rangle + \lambda\langle y, x\rangle + \lambda\langle y, \lambda y\rangle\\
&= \|x\|^2 + \bar{\lambda}\langle x, y\rangle + \lambda\overline{\langle x, y\rangle} + \lambda\bar{\lambda}\|y\|^2\\
&= \|x\|^2 + 2\Re\big(\bar{\lambda}\langle x, y\rangle\big) + |\lambda|^2\|y\|^2.
\end{aligned}$$

If $\langle x, y\rangle = 0$ the Cauchy - Schwarz inequality is trivial, so assume that $\langle x, y\rangle \neq 0$. Let $\lambda := tu$ with $u := \frac{\langle x,y\rangle}{|\langle x,y\rangle|}$, so that

$$\bar{\lambda}\langle x, y\rangle = t\frac{\overline{\langle x, y\rangle}\langle x, y\rangle}{|\langle x, y\rangle|} = t|\langle x, y\rangle| \qquad \text{and} \qquad |\lambda|^2 = t^2.$$

Hence,

$$0 < \|x\|^2 + 2t|\langle x, y\rangle| + t^2\|y\|^2 = \Big(\|y\|t + \frac{|\langle x, y\rangle|}{\|y\|}\Big)^2 + \|x\|^2 - \Big(\frac{|\langle x, y\rangle|}{\|y\|}\Big)^2.$$

By choosing $t = -|\langle x, y\rangle|/\|y\|^2$, we obtain that

$$\frac{|\langle x, y\rangle|^2}{\|y\|^2} < \|x\|^2,$$

which proves the assertion.

℘ **Inner-product spaces are normed**
If $(X, \langle\cdot,\cdot\rangle)$ is an inner-product space, then $\|x\| = \langle x, x\rangle^{1/2}$ defines a norm on X.

Proof
Positive homogeneity:

$$\|\lambda x\| = \langle \lambda x, \lambda x\rangle^{1/2} = \big(\lambda\bar{\lambda}\langle x, x\rangle\big)^{1/2} = \big(|\lambda|^2\|x\|^2\big)^{1/2} = |\lambda|\|x\|.$$

Triangle inequality: By the Cauchy - Schwarz inequality,

$$\begin{aligned}\|x + y\|^2 &= \|x\|^2 + 2\Re\langle x, y\rangle + \|y\|^2\\ &\leq \|x\|^2 + 2|\langle x, y\rangle| + \|y\|^2\\ &\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2\\ &= \big(\|x\| + \|y\|\big)^2.\end{aligned}$$

Non-degeneracy:

$$\|x\| = 0 \quad\Longleftrightarrow\quad \|x\|^2 = 0 \quad\Longleftrightarrow\quad \langle x, x\rangle = 0 \quad\Longleftrightarrow\quad x = 0,$$

according to the positive definiteness of the inner product.

℘ **Parallelogram law and polarization identity**
Let $(X, \|\cdot\|)$ be a normed space. Then the **parallelogram law**

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

holds exactly if $\|\cdot\| = \langle\cdot,\cdot\rangle^{1/2}$ can be defined using an inner product on X. If so,

$$\langle x, y\rangle = \frac{1}{4}\big(\|x + y\|^2 - \|x - y\|^2\big),$$

if X is real, and

$$\langle x, y\rangle = \frac{1}{4}\sum_{k=0}^{3} i^k\|x + i^k y\|^2,$$

if X is complex.

Proof
We only show that the parallelogram law and polarization identity hold in an inner product space; the other direction (starting with a norm and the parallelogram identity to define an inner product) is left as an exercise.

Parallelogram law: If X is an inner-product space, then

$$\|x \pm y\|^2 = \|x\|^2 \pm 2\Re\langle x, y\rangle + \|y\|^2;$$

the parallelogram law follows from adding these two equations to each other.

Polarization identity: When X is a real inner-product space, it follows directly that

$$\|x + y\|^2 - \|x - y\|^2 = \big(\|x\|^2 + 2\langle x, y\rangle + \|y\|^2\big) - \big(\|x\|^2 - 2\langle x, y\rangle + \|y\|^2\big) = 4\langle x, y\rangle.$$

If X is complex, the corresponding calculation yields that

$$\begin{aligned}\sum_{k=0}^{3} i^k \|x + i^k y\|^2 &= \sum_{k=0}^{3} i^k \big(\|x\|^2 + 2\Re\langle x, i^k y\rangle + \|i^k y\|^2\big) \\ &= \big(\|x\|^2 + 2\Re\langle x, y\rangle + \|y\|^2\big) - \big(\|x\|^2 - 2\Re\langle x, y\rangle + \|y\|^2\big) \\ &\quad + i\big(\|x\|^2 - 2\Re i\langle x, y\rangle + \|y\|^2\big) - i\big(\|x\|^2 + 2\Re i\langle x, y\rangle + \|y\|^2\big).\end{aligned}$$

Since $\Re iz = -\Im z$ for any $z \in \mathbb{C}$, we obtain

$$\sum_{k=0}^{3} i^k \|x + i^k y\|^2 = 4\Re\langle x, y\rangle + 4\Im\langle x, y\rangle = 4\langle x, y\rangle.$$

Ex.

- **Pythagoras' theorem**: If $\langle x, y\rangle = 0$ in an inner-product space, then

  $$\|x + y\|^2 = \|x\|^2 + \|y\|^2,$$

  which, in $\mathbb{R}^2$, we recognize as

  $$a^2 + b^2 = c^2,$$

  with a, b, c the sides of a right-angled triangle.
- If we define $\langle x, y\rangle := \frac{1}{4}\left(\|x + y\|^2 - \|x - y\|^2\right)$ in $\mathbb{R}^2$ using the polarization identity , we see that

  $$\begin{aligned}\langle x, y\rangle &= \frac{1}{4}\left((x_1 + y_1)^2 + (x_2 + y_2)^2\right) - \frac{1}{4}\left((x_1 - y_1)^2 + (x_2 - y_2)^2\right) \\ &= \frac{1}{4}\left(x_1^2 + 2x_1y_1 + y_1^2 + x_2^2 + 2x_2y_2 + y_2^2\right) - \frac{1}{4}\left(x_1^2 - 2x_1y_1 + y_1^2 + x_2^2 - 2x_2y_2 + y_2^2\right) \\ &= x_1y_1 + x_2y_2\end{aligned}$$

  is the standard dot product.

3.6 Closest point theorem

Hilbert space

A complete inner-product space is called a **Hilbert space**. Similarly, inner-product spaces are sometimes called **pre-Hilbert spaces**.

Ex.

- The Banach spaces $\mathbb{R}^n$, $l_2(\mathbb{R})$ and $L_2(I,\mathbb{R})$, as well as their complex counterparts $\mathbb{C}^n$, $l_2(\mathbb{C})$ and $L_2(I,\mathbb{C})$, all have norms that come from inner products:

$$\langle x,y\rangle_{\mathbb{C}^n} = \sum_{j=1}^{n} x_j\bar{y_j} \quad \text{in} \quad \mathbb{C}^n, \qquad \langle x,y\rangle_{l_2} = \sum_{j=1}^{\infty} x_j\bar{y_j} \quad \text{in} \quad l_2,$$

  and

$$\langle x,y\rangle_{L_2} = \int_I x(s)\overline{y(s)}\,ds \quad \text{in} \quad L_2.$$

  (If the spaces are real, there are no complex conjugates.)
- Thus, they are all Hilbert spaces. In particular, this proves the l_2- and L_2-norms defined earlier in this course are indeed norms.
- The space of bounded continuous functions on a finite open interval, $BC((a,b),\mathbb{R})$ or $BC((a,b),\mathbb{C})$, can be equipped with the L_2-inner product. This is a pre-Hilbert space, the completion of which is $L_2((a,b),\mathbb{R})$ ($L_2((a,b),\mathbb{C})$).

Convex sets and the closest point property

Let X be a linear space. A subset $M \subset X$ is called **convex** if

$$x,y \in M \quad \Longrightarrow \quad tx+(1-t)y \in M \quad \text{for all} \quad t \in (0,1),$$

i.e., if all points in M can be joined by line segments in M.

Ex.

- Any **hyperbox** $\{x \in \mathbb{R}^n : a_j \le x_j \le b_j\}$ is convex.
- Intuitively, any region with a 'hole', like $\mathbb{R}^n \setminus B_1$, is *not* convex.
- Linear subspaces are convex:

$$x,y \in M \quad \Longrightarrow \quad \mu x+\lambda y \in M \quad \text{for all scalars } \mu,\lambda,$$

  clearly implies that $tx+(1-t)y \in M$ for all $t \in (0,1)$.

℘ **Closest point property (Minimal distance theorem)**

Let H be a Hilbert space, and $M \subset H$ a non-empty, closed and convex subset of H. For any $x_0 \in H$ there is a unique element $y_0 \in M$ such that

$$\|x_0-y_0\| = \inf_{y\in M}\|x_0-y\|.$$

N.b. The number $\inf_{y\in M}\|x_0-y\|$ is the **distance from** x_0 **to** M, denoted $\operatorname{dist}(x_0,M)$.

Proof

A minimizing sequence: Since $M \neq \emptyset$, the number $d := \inf_{y\in M}\|x_0-y\|$ is finite and non-negative, and by the definition of infinimum, there exists a minimizing sequence $\{y_j\}_{j\in\mathbb{N}} \subset M$ such that

$$\lim_{j\to\infty}\|x_0-y_j\| = d.$$

$\{y_j\}_{j\in\mathbb{N}}$ **is Cauchy**: By the parallelogram law applied to x_0-y_n, x_0-y_m, we have

$$\|2x_0-(y_m+y_n)\|^2+\|y_m-y_n\|^2 = 2\|x_0-y_m\|^2+2\|x_0-y_n\|^2 \to 4d^2, \qquad m,n\to\infty.$$

In view of that M is convex and d minimal, we also have that

$$\|2x_0 - (y_m + y_n)\|^2 = 4\left\|x_0 - \frac{y_m + y_n}{2}\right\|^2 \geq 4d^2.$$

Consequently,

$$\|y_m - y_n\|^2 \to 0 \quad \text{as} \quad m, n \to \infty.$$

Since $M \subset H$ is closed and H is complete, there exists

$$y_0 = \lim_{j\to\infty} y_j \in M \quad \text{with} \quad \|x_0 - y_0\| = \lim_{j\to\infty} \|x_0 - y_j\| = d.$$

Uniqueness: Suppose that $z_0 \in M$ satisfies $\|x_0 - z_0\| = d$. Then $\frac{y_0+z_0}{2} \in M$ and the parallelogram law (applied to $x_0 - y_0$, $x_0 - z_0$) yields that

$$\|y_0 - z_0\|^2 = 2\|x_0 - y_0\|^2 + 2\|x_0 - z_0\|^2 - 4\left\|x_0 - \frac{y_0 + z_0}{2}\right\|^2 \leq 2d^2 + 2d^2 - 4d^2 = 0,$$

so that $z_0 = y_0$.

Ex.

- In the Hilbert space $\mathbb{R}^2$:
  - The closed unit disk $\{x_1^2 + x_2^2 \leq 1\}$ contains a unique element that minimizes the distance to the point $(2, 0)$ (namely $(1, 0)$).
  - The subgraph $\{x_2 \leq x_1^2\}$ is closed but not convex; it has more than one point minimizing the distance to the point $(0, 1)$.
  - The open unit ball $\{x_1^2 + x_2^2 < 1\}$ is convex but not closed; it has no element minimizing the distance to a point outside itself.
- The same proof as above gives a slight modification of the closest point theorem that can be useful in some examples. Suppose that V is an inner-product space and M is a convex closed subset that is complete in the metric induced by the metric in V then for any $x \in V$ there exists $y_0 \in M$ such that $\|x - y_0\| = \inf_{y\in M} \|x - y\|$.
- Let

  $$M_n := \operatorname{span}\{e^{ikx}\}_{k=-n}^{n}$$

  be the closed linear span of trigonometric functions

  $$1, e^{ix}, e^{-ix} \ldots, e^{inx}, e^{-inx} \in C((-\pi, \pi), \mathbb{C})$$

  with the inner-product $\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)\overline{g(x)}dx$. Then M_n is complete metric space. For any $n \in \mathbb{N}$ and any $f \in C((-\pi, \pi), \mathbb{C})$ there is a unique linear combination of $1, e^{ix}, e^{-ix} \ldots, e^{inx}, e^{-inx}$ that minimizes the L_2-distance to f:

  $$\int_{-\pi}^{\pi} \left|f(x) - \sum_{k=-n}^{n} c_k e^{ikx}\right|^2 dx = \min_{g\in M_n} \int_{-\pi}^{\pi} \left|f(x) - g(x)\right|^2 dx.$$

  The coefficients c_k are known as (complex) **Fourier coefficients** of the function f. The same statement holds for any $f \in L_2((-\pi, \pi), \mathbb{C})$ which is the completion of $C((-\pi, pi), \mathbb{C})$ in L^2-norm.

3.7 Orthogonality

Consider an inner-product space $(X, \langle\cdot,\cdot\rangle)$ over a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. When X is complete, we shall write H to indicate that it is a Hilbert space.

The projection theorem

Orthogonal vectors and set

- Two vectors $x, y \in X$ are said to be **orthogonal**,

$$x \perp y \quad \overset{\text{def}}{\iff} \quad \langle x, y\rangle = 0.$$

- A vector $x \in X$ is said to be **orthogonal** to a set $M \subset X$,

$$x \perp M \quad \overset{\text{def}}{\iff} \quad \langle x, y\rangle = 0 \quad \text{for all } y \in M.$$

- Two sets $M, N \subset X$ are said to be **orthogonal**,

$$M \perp N \quad \overset{\text{def}}{\iff} \quad \langle x, y\rangle = 0 \quad \text{for all} \quad x \in M, y \in N.$$

- The **orthogonal complement** of a set $M \in X$ consists of all vectors orthogonal to M:

$$M^{\perp} \quad \overset{\text{def.}}{=} \quad \{x \in X \colon x \perp M\}.$$

The word **perpendicular** is sometimes used interchangeably with 'orthogonal', but mostly in $\mathbb{R}^n$.

Ex.

- In $\mathbb{R}^3$, the vector $(1, 2, 1)$ is orthogonal to the plane $\{x_1 + 2x_2 + x_3 = 0\}$.
- In $L_2((-\pi, \pi), \mathbb{C})$ the vectors e^{ikx}, $k \in \mathbb{Z}$, are all orthogonal to each other:

$$\langle e^{ik_1 x}, e^{ik_2 x}\rangle = \int_{-\pi}^{\pi} e^{ik_1 x}\overline{e^{ik_2 x}}\, dx = \int_{-\pi}^{\pi} e^{i(k_1 - k_2)x}\, dx = \frac{e^{i(k_1-k_2)x}}{i(k_1 - k_2)}\bigg|_{-\pi}^{\pi} = 0 \quad \text{for} \quad k_1 \neq k_2,$$

  by periodicity of $e^{ikx} = \cos(kx) + i\sin(kx)$.
- In $l_2(\mathbb{R})$, with $e_1 = (1, 0, 0, \ldots)$,

$$\{e_1\}^{\perp} = \{x \in l_2(\mathbb{R}) \colon x = (0, x_2, x_3, \ldots)\}.$$

If $X, Y \subset V$ are subspaces of a vector space V, then we write

$$X \oplus Y = V \quad \iff \quad X \cap Y = \{0\} \quad \text{and} \quad X + Y \overset{\text{def.}}{=} \{x + y \colon x \in X, y \in Y\} = V.$$

More precisely, the **direct sum** $X \oplus Y$ of two vector spaces (both real, or both complex) is the space of pairs (x, y) with the naturally induced vector addition and scalar multiplication:

$$X \oplus Y \overset{\text{def.}}{=} \{(x, y) \in X \times Y\},$$

where

$$(x_1, y_1) + (x_2, y_2) \overset{\text{def.}}{=} (x_1 + x_2, y_1 + y_2) \quad \text{and} \quad \lambda(x, y) \overset{\text{def.}}{=} (\lambda x, \lambda y).$$

and the equality $X \oplus Y = V$ should be interpreted in terms of isomorphisms (V can be represented as $X \oplus Y$).

Note that if X and Y are finite dimensional then

$$\dim(X \oplus Y) = \dim(X) + \dim(Y).$$

If $\{e_j\}_j$ is a basis for X and $\{b_k\}_k$ is a basis for Y then $\{e_j\} \cup \{b_k\}$ is a basis for $X \oplus Y$.

Ex.

$$\mathbb{R}^3 = \text{span}\{(1,0,0)\} \oplus \text{span}\{(0,1,0),(0,0,1)\},$$

but also

$$\mathbb{R}^3 = \text{span}\{(1,2,1)\} \oplus \{x_1 + 2x_2 + x_3 = 0\}.$$

℘ **The projection theorem**

Let $M \subset H$ be a closed linear subspace of a Hilbert space H. Then $H = M \oplus M^\perp$.

Proof

Existence of $y_0 \in M$: Pick $x_0 \in H$. By the minimal distance theorem, there exists a unique point $y_0 \in M$ with

$$\|x_0 - y_0\| = \inf_{y \in M} \|x_0 - y\|.$$

Existence of $x_0 - y_0 \in M^\perp$: Since M is a subspace, $y_0 + \lambda y \in M$ for any $y \in M$, $\lambda \in \mathbb{K}$. Hence

$$\|x_0 - y_0\|^2 \leq \|x_0 - y_0 - \lambda y\|^2 = \|x_0 - y_0\|^2 - 2\Re(\lambda\langle y, x_0 - y_0\rangle) + |\lambda|^2\|y\|^2,$$

and

$$-2\Re(\lambda\langle y, x_0 - y_0\rangle) + |\lambda|^2\|y\|^2 \geq 0.$$

By taking $\lambda = \varepsilon \ll 1$, we see that

$$\Re(\lambda\langle y, x_0 - y_0\rangle) \leq 0,$$

and, similarly, by taking $\lambda = -i\varepsilon$, that

$$\Im(\lambda\langle y, x_0 - y_0\rangle) \leq 0.$$

Since $y \in M$ is arbitrary, by exchanging $-y$ for y, we obtain

$$\langle y, x_0 - y_0\rangle = 0 \quad \text{for any} \quad y \in M.$$

Thus we can write

$$x_0 = y_0 + (x_0 - y_0), \quad \text{where} \quad y_0 \in M, \quad x_0 - y_0 \in M^\perp.$$

Uniqueness: If we have two representations $x_0 = y_0 + z_0$ and $x_0 = \tilde{y}_0 + \tilde{z}_0$, then

$$M \ni y_0 - \tilde{y}_0 = \tilde{z}_0 - z_0 \in M^\perp,$$

but only the zero vector is orthogonal to itself, implying that $y_0 = \tilde{y}_0$ and $z_0 = \tilde{z}_0$.

Orthogonal projection

Let M be a closed linear subspace of H, the projection theorem says that for any $x \in H$ there exists a unique $y \in M$ such that $x - y \perp M$, for this case y is called the **orthogonal projection** of x into M.

Ex. Let $x = (1, 3) \in \mathbb{R}^2$ and $u = (2, -1)$. Consider $M = \text{span}(u) = \{(2t, -t) : t \in \mathbb{R}\}$. Then $x = p + q$, where $p \in M$ is the orthogonal projection of x onto u (or the line generated by u) and q is orthogonal to u. Then p is

$$p = \frac{\langle x, u\rangle}{\langle u, u\rangle} u.$$

For the given x and u we get

$$p = \frac{-1}{5}(2, -1) = (0.4, -0.2).$$

Corollary: strict subspace characterization
If $M \subsetneq H$ is a closed linear subspace of H, there exists a non-zero vector $z_0 \in H$ with $z_0 \perp M$.

Proof
Since $M \neq H$ there exists $x_0 \in H \setminus M$. According to the projection theorem, $x_0 = y_0 + z_0$ with $y_0 \in M$, $z_0 \in M^\perp$. Then $z_0 \neq 0$, and $z_0 \perp M$ is the vector we are looking for.

Ex.

- Let $M = \overline{l_0}$ be the closure of

$$l_0 = \{x \in l_2 \colon \{x_j\}_{j\in\mathbb{N}} \text{ has finitely many non-zero entries}\}$$

in l_2. Is $M = l_2$? Say there exists $z \in l_2$ such that $z \perp M$. Since $\{e_j\}_{j\in\mathbb{N}} \subset M$, we have

$$\langle z, e_j\rangle = z_j = 0 \quad \text{for all} \quad j \in \mathbb{N}.$$

Thus $z = 0$, and $\overline{l_0} = l_2$.

Matrices: null spaces, column spaces and row spaces

Let $A = (a_{ij})_{ij} \in M_{m\times n}(\mathbb{R})$ be a matrix, $A = (A_1, ..., A_n)$, where A_j are column-vector of A, $A_j \in \mathbb{R}^m, 1 \leq j \leq n$. The **transpose** of A is an $n \times m$ matrix with rows $A_1, ...A_n$, $A^t = (a_j i)_{ij}$.

The **null space of** A also called the kernel of A is the subspace of solutions to $Ax = \mathbf{0}$:

$$\begin{aligned} x \in \ker(A) \quad &\iff \quad Ax = 0 \quad \iff \quad \sum_{j=1}^{n} a_{ij}x_j = 0 \quad \forall i = 1, \dots, m \\ &\iff \quad (x_1, \dots, x_n) \perp (a_{i1}, \dots, a_{in}) \quad \text{for all } i = 1, \dots, n. \end{aligned}$$

Thus, *the kernel is the space of vectors $x \in \mathbb{R}^n$ which are orthogonal to the row vectors of A.*

The **column space of** A is the range of A: since

$$Ax = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix} + \ldots + x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix},$$

we have that

$$\text{ran}(A) = \{Ax \colon x \in \mathbb{R}^n\} = \left\{ \sum_{j=1}^{n} x_j A_j \colon (x_1, \dots, x_n) \in \mathbb{R}^n \right\} = \text{span}\{A_1, \dots, A_n\}$$

is the subspace of $\mathbb{R}^m$ spanned by the column vectors A_j, $j = 1, \dots, n$, of A.

The **row space of** A to be the space spanned by the row vectors of A. Then

$$\text{row space of } A = \text{ column space of } A^t,$$

where $A^t = (a_{ji})$ is the transpose of $A = (a_{ij})$. The dimension of the row space is called the **rank** of A

℘ **The kernel of a matrix is perpendicular to the range of its transpose**
Let $A \in M_{m\times n}(\mathbb{R})$. Then

$$\ker(A) = (\text{ran}(A^t))^{\perp},$$

Proof
As shown above, a vector is in the null space of A if and only if it is perpendicular to all rows of A, it means that it is perpendicular to the row space of A. The row space of A equals the column space of A^t (this is the definition of the matrix transpose). The proposition follows.

℘ **Rank–nullity theorem for matrices**
Let $A \in M_{m\times n}(\mathbb{R})$. Then

$$\mathbb{R}^n = \ker(A) \oplus \text{ran}(A^t), \quad n = \dim(\ker(A)) + \dim(\text{ran}(A^t)).$$

N.b. The name comes from that $\dim\ker(A)$ is the **nullity of** A. Thus, the sum of the rank and the nullity of A equals the dimension of its ground space (domain). The first statement

$$\mathbb{R}^n = \ker(A) \oplus \text{ran}(A^t)$$

is called the geometric interpretation of the rank-nullity theorem.

3.8 Orthogonal sets and bases

Orthonormal systems

A sequence $\{e_j\}_{j\in\mathbb{N}}$ of non-zero elements in an inner-product space is called **orthogonal** if $e_j \perp e_k$ for $j \neq k$. If, in addition, $\|e_j\| = 1$ for all $j \in \mathbb{N}$, it is called **orthonormal**.

Ex.
- The canonical basis $\{e_j\}_j$ is an orthonormal basis in $\mathbb{R}^n$, $\mathbb{C}^n$ and l_2 (real or complex).
- **Generalized Pythagoras' theorem** Suppose that $\{v_j\}_{j=1}^n$ is an orthogonal system, then
$$\|v_1 + v_2 ... + v_n\|^2 = \|v_1\|^2 + \|v_2\|^2 + ... + \|v_n\|^2.$$
It follows from the usual Pythagoras theorem by induction.
- The sequence $\{\frac{1}{\sqrt{2\pi}}e^{ikx}\}_{k\in\mathbb{Z}}$ is an orthonormal sequence in $L_2((-\pi,\pi),\mathbb{C})$, since

$$\left\|\frac{1}{\sqrt{2\pi}}e^{ikx}\right\| = \left(\int_{-\pi}^{\pi}\frac{1}{\sqrt{2\pi}}e^{ikx}\overline{\frac{1}{\sqrt{2\pi}}e^{ikx}}\,dx\right)^{1/2} = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{ikx}e^{-ikx}\,dx\right)^{1/2} = 1.$$

Orthogonal projection onto the span of an orthonormal system
If $\{e_1,\ldots,e_n\}$ is an orthonormal system, then $y = \sum_{j=1}^n\langle x, e_j\rangle e_j$ is the closest point to x in $\text{span}\{e_1,\ldots,e_n\}$, with $d = \|x - y\|$ given by

$$d^2 = \|x\|^2 - \sum_{j=1}^{n}|\langle x, e_j\rangle|^2.$$

N.b. In particular, if $x \in \text{span}\{e_1,\ldots,e_n\}$, then $x = \sum_{j=1}^n\langle x, e_j\rangle e_j$.

Proof
Let $y = \sum c_j e_j$ be the closest point to x in $M = \text{span}\{e_1, ..., e_n\}$. Then by the orthogonal projection theorem $x = y + y_1$, where $y_1 \in M^\perp$. Then for each $j = 1, ..., n$ we have $\langle x, e_j\rangle = \langle y, e_j\rangle = c_j$. Thus $y = \sum_{j=1}^n \langle x, e_j\rangle e_j$. Further the Pythagoras theorem implies

$$\|x - y\|^2 = \|x\|^2 - \|y\|^2,$$

since $x - y = y_1 \perp y$. Finally, the generalized Pythagoras theorem applied to $y = \sum_j \langle x, e_j\rangle e_j$, gives

$$\|x - y\|^2 = \|x\|^2 - \sum_{j=1}^n |\langle x, e_j\rangle|^2.$$

Corollary: Fourier coefficients are best possible coefficients
Let $\{e_j\}_{j=1}^N$ be an orthonormal sequence. Then for any $x \in X$ and any scalars $\lambda_1, \ldots, \lambda_N \in \mathbb{K}$

$$\Big\|x - \sum_{j=1}^N \lambda_j e_j\Big\| \geq \Big\|x - \sum_{j=1}^N \langle x, e_j\rangle e_j\Big\|.$$

The equality holds if and only if $\lambda_j = \langle x, e_j\rangle$ for all $j \in \mathbb{N}$.

Ex. In $\mathbb{R}^3$, what is the closest point in the plane spanned by $e_1 := \frac{1}{\sqrt{2}}(1, 1, 0)$ and $e_2 := (0, 0, 1)$ to the point $x = (2, 1, 1)$? We have

$$\begin{aligned}\langle x, e_1\rangle e_1 + \langle x, e_2\rangle e_2 &= \big((2, 1, 1) \cdot \tfrac{1}{\sqrt{2}}(1, 1, 0)\big)\tfrac{1}{\sqrt{2}}(1, 1, 0) + \big((2, 1, 1) \cdot (0, 0, 1)\big)(0, 0, 1) \\ &= \tfrac{3}{2}(1, 1, 0) + (0, 0, 1) = (\tfrac{3}{2}, \tfrac{3}{2}, 1).\end{aligned}$$

The distance is

$$\big(\|x\|^2 - |\langle x, e_1\rangle|^2 - |\langle x, e_2\rangle|^2\big)^{1/2} = \big(6 - \tfrac{9}{2} - 1\big)^{1/2} = \frac{1}{\sqrt{2}},$$

which can be checked to fit with $|(2, 1, 1) - (\frac{3}{2}, \frac{3}{2}, 1)|$.

℘ **Bessel's inequality**
An orthonormal sequence satisfies $\sum_{j\in\mathbb{N}} |\langle x, e_j\rangle|^2 \leq \|x\|^2$, for all $x \in X$.

Proof
Let $x \in X$, first for any finite N

$$\sum_{j=1}^N |\langle x, e_j\rangle|^2 \leq \|x\|^2$$

by the inequality above. Since this is true for any partial sum, it also holds for the sum of the series

$$\sum_j |\langle x, e_j\rangle|^2 = \lim_{N\to\infty} \sum_{j=1}^N |\langle x, e_j\rangle|^2.$$

Convergence as an l_2-property (in Hilbert spaces)
Let $\{e_j\}_{j\in\mathbb{N}}$ be an orthonormal sequence in a Hilbert space H, and $\{\lambda_j\}_{j\in\mathbb{N}}$ a sequence of scalars. Then

$$\exists \lim_{N\to\infty} \sum_{j=1}^N \lambda_j e_j \quad \text{in } H \quad \Longleftrightarrow \quad \sum_{j=1}^\infty |\lambda_j|^2 < \infty.$$

In that case, $\| \sum_{j\in\mathbb{N}} \lambda_j e_j\|^2 = \sum_{j\in\mathbb{N}} |\lambda_j|^2$.

N.b. A consequence of this is that every infinite-dimensional separable Hilbert space can be identified with l_2. If the Hilbert space is finite, it can be identified with $\mathbb{R}^n$ or $\mathbb{C}^n$; if it is not separable, it is bigger than l_2.

Proof

Let $x_n := \sum_{j=1}^n \lambda_j e_j$. For $m > n$,

$$\|x_m - x_n\|^2 = \Big\| \sum_{j=n+1}^m \lambda_j e_j \Big\|^2 = \sum_{j,k=n+1}^m \lambda_j \overline{\lambda_k} \langle e_j, e_k\rangle = \sum_{j=n+1}^m |\lambda_j|^2,$$

meaning that $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy exactly if $\sum_{j=1}^\infty |\lambda_j|^2$ converges in $\mathbb{R}$. Since H is complete, this happens exactly if $\{x_n\}_{n\in\mathbb{N}}$ converges in H. A similar calculation shows that

$$\Big\| \sum_{j=1}^m \lambda_j e_j \Big\|^2 = \sum_{j=1}^m |\lambda_j|^2.$$

When (one of) these sums converge we may let $m \to \infty$ to obtain the desired equality.

℘ **Orthogonal projection and the closest point**

If $\{e_j\}_j$ is an orthonormal sequence, then the closest point to x in $M = \overline{\text{span}\{e_j\}_j}$ is the orthogonal projection of x onto M that is given by $y = \sum_j \langle x, e_j\rangle e_j$.

Proof

Let $y = \sum_{j=1}^\infty \langle x, e_j\rangle e_j$, the series converges by since $\sum_j |\langle x, e_j\rangle|^2 < +\infty$ by the Bessel inequality.

$$\langle y, e_k\rangle = \sum_{j=1}^\infty \langle x, e_j\rangle \langle e_j, e_k\rangle = \langle x, e_k\rangle.$$

Thus $x - y$ is orthogonal to each e_k and therefore to M. Hence y is the orthogonal projection of x onto M and hence it is the closest point.

Ex. Often minimization problems are problems about the orthogonal projections. For example, if we want to find $a, b \in \mathbb{C}$ such that

$$\int_{-\pi}^{\pi} |t - ae^{-it} - be^{it}|^2 dt$$

is minimal, we are asking about the projection of the function $f(t) = t$ onto $V = \text{span}(e^{-it}, e^{it})$ in $L^2((-\pi, \pi), \mathbb{C})$. The functions $\{e^{-it}, e^{it}\}$ form an orthogonal basis for V. Then the projection of f onto V is given by

$$\frac{\langle f, e^{-it}\rangle}{\langle e^{-it}, e^{-it}\rangle} e^{-it} + \frac{\langle f, e^{it}\rangle}{\langle e^{it}, e^{it}\rangle} e^{it}.$$

Using integration by parts, we get

$$\langle f, e^{-it}\rangle = \int_{-\pi}^{\pi} te^{it} dt = 2\pi i, \quad \langle f, e^{it}\rangle = \int_{-\pi}^{\pi} te^{-it} dt = -2\pi.$$

Also, $\|e^{it}\|^2 = \|e^{-it}\|^2 = 2\pi$. Then $a = i,\ b = -i$.

Orthonormal bases

An orthonormal system $\{e_j\}_j \subset H$ is called an **orthonormal basis** for H if

$$x = \sum_j \langle x, e_j \rangle e_j \quad \text{for all } x \in H.$$

It is not difficult to see that an orthonormal system in a Hilbert space is an orthonormal basis if and only if it is a Schauder basis.

Ex.

- In $\mathbb{R}^n$, $\mathbb{C}^n$, $l_2(\mathbb{R})$ and $l_2(\mathbb{C})$, the canonical basis $\{e_j\}_j$ is also an orthonormal basis.
- The vectors

  $$\tfrac{1}{\sqrt{2}}(1,1,0), \quad \tfrac{1}{\sqrt{2}}(1,-1,0), \quad (0,0,1)$$

  form an orthonormal basis for $\mathbb{R}^3$.
- $\{\frac{1}{\sqrt{2}}, \cos(x), \sin(x), \cos(2x), \sin(2x), \ldots\}$ is an orthonormal basis for $L_2((-\pi,\pi),\mathbb{R})$ if we equip it with the inner product

  $$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) g(x)\, dx.$$

  (One may also use the standard inner product and scale the functions with $1/\sqrt{\pi}$.)
- $\{e^{ikx}\}_{k\in\mathbb{Z}}$ is an orthonormal basis for $L_2((-\pi,\pi),\mathbb{C})$ if we equip it with the inner product

  $$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \overline{g(x)}\, dx.$$

  Equivalently, one may use the standard inner product and scale the functions with $1/\sqrt{2\pi}$.

℘ **The Fourier series theorem**

Let $M = \{e_j\}_{j\in\mathbb{N}}$ be an orthonormal sequence in a Hilbert space H. Then the following are equivalent:

- M is a complete sequence, i.e., there is no non-zero $y \in M$ such that $y \perp e_j$ for each j,
- $\overline{\text{span}(M)} = H$,
- M is an orthonormal basis for H,
- For all $x \in H$, $\|x\|^2 = \sum_{j\in\mathbb{N}} |\langle x, e_j \rangle|^2$.

N.b.

- An analog result holds for orthonormal systems (in particular: for finite sets).
- The last equality is known as **Parseval's identity**.

Proof

(i) $\Longrightarrow$ **(ii)**: If M is complete, then $M^\perp = \{0\}$, so that $\overline{\text{span}(M)} = H$ (else, there would exists a non-zero vector in its orthogonal complement.

(ii) $\Longrightarrow$ **(iii)**: If $\overline{\text{span}(M)} = H$, then, for any $x \in H$, there exist $\{\lambda_j\}_{j\in\mathbb{N}}$ such that

$$\lim_{N\to\infty} \sum_{j=1}^{N} \lambda_j e_j = x.$$

But

$$\Big\| \sum_{j=1}^{N} \lambda_j e_j - x \Big\|^2 \geq \Big\| \sum_{j=1}^{N} \langle x, e_j \rangle e_j - x \Big\|^2 \geq 0,$$

so that $x = \sum_{j=1}^{\infty} \langle x, e_j \rangle e_j$.

(iii) $\Longrightarrow$ (iv): If M is an orthonormal basis, it is immediate that

$$\|x\|^2 = \Big\langle \sum_{j\in\mathbb{N}} \langle x, e_j \rangle e_j, \sum_{j\in\mathbb{N}} \langle x, e_j \rangle e_j \Big\rangle = \sum_{j\in\mathbb{N}} |\langle x, e_j \rangle|^2.$$

(iv) $\Longrightarrow$ (i): Finally, if $\|x\|^2 = \sum_{j\in\mathbb{N}} |\langle x, e_j \rangle|^2$ for all $x \in H$, and $x \perp M$, then $\|x\| = 0$. Hence, there is no non-zero vector in $M^{\perp}$, which is the definition of M being complete.

Ex.

- Consider $L_2((-\pi, \pi), \mathbb{C})$ with the orthonormal basis $\{e^{ikx}\}_{k\in\mathbb{Z}}$ and the inner product

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \overline{g(x)}\, dx.$$

The Fourier coefficients are given by

$$\hat{f}_k := \langle f, e^{ik\cdot} \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx}\, dx,$$

and Parseval's identity states that

$$\|f\|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2\, dx = \sum_{k=-\infty}^{\infty} |\hat{f}_k|^2 = \sum_{k=-\infty}^{\infty} |\langle f, e^{ik\cdot} \rangle|^2.$$

The Gram–Schmidt orthogonalization

A set $\{v_1, \ldots, v_n\} \subset H$ of linearly independent vectors in $\mathbb{R}^m$ can be transformedinto an orthonormal system using the **Gram–Schmidt orthogonalization** algorithm.[1] Define

$$e_1 := \frac{v_1}{\|v_1\|},$$

$$\tilde{e}_2 := v_2 - \langle v_2, e_1 \rangle e_1, \qquad e_2 := \frac{\tilde{e}_2}{\|\tilde{e}_2\|},$$

and, recursively,

$$\tilde{e}_{k+1} := v_{k+1} - \sum_{j=1}^{k} \langle v_{k+1}, e_j \rangle e_j, \qquad e_{k+1} := \frac{\tilde{e}_{k+1}}{\|\tilde{e}_{k+1}\|}, \qquad k = 1, \ldots, n-1.$$

Then $\{e_1, \ldots, e_n\}$ is an orthonormal system.

QR-decompositions

If, in the Gram–Schmidt orthoghonalization, we express the vectors $\{v_1, \ldots, v_n\}$ in terms of $\{e_1, \ldots, e_n\}$, we obtain

$$v_1 = \langle v_1, e_1 \rangle e_1, \qquad v_2 = \langle v_2, e_1 \rangle e_1 + \langle v_2, e_2 \rangle e_2, \qquad v_k = \sum_{j=1}^{k} \langle v_k, e_j \rangle e_j,$$

[1] The Gram–Schmidt orthogonalization is equally valid for linearly independent sequences.

which can be seen either by direct calculation or from the 'closest point'-corollary ((note taht $\text{span}\{v_1, \dots v_k\} = \text{span}\{e_1, \dots e_k\}$ for $k = 1, \dots, n$). This gives us the QR-decomposition of a full-rank matrix A:

If $A = [v_1, \dots, v_n] \in M_{m\times n}(\mathbb{R})$ is matrix of rank m (so that its column vectors are linear independent in $\mathbb{R}^m$), the Gram–Schmidt orthogonalization applied to the columns vectors $v_1, \dots, v_n$ yields the QR-decomposition of A:

$$A = QR = \begin{bmatrix} e_1, \cdots, e_n \end{bmatrix} \begin{bmatrix} \langle v_1, e_1\rangle & \langle v_2, e_1\rangle & \dots & \langle v_n, e_1\rangle \\ 0 & \langle v_2, e_2\rangle & \dots & \langle v_n, e_2\rangle \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & \langle v_n, e_n\rangle \end{bmatrix},$$

where $Q = [e_1, ..., e_n] \in M_{m\times n}(\mathbb{R})$ and $R \in M_{n\times n}(\mathbb{R})$ is upper (right) triangular. The columns of Q are pairwise orthogonal and has norms one, thus $Q^tQ = I$. To summarize

$\wp$ **QR-decomposition**
Any matrix $A \in M_{m\times n}(\mathbb{R})$ with linearly independent columns can be factorized into $A = QR$, where $Q \in M_{m\times n}(\mathbb{R})$ has orthonormal columns and $R \in M_{n\times n}(\mathbb{R})$ is upper-triangular and invertible.

For the case of a *square* non-singular matrix A the first factor in the decomposition $A = QR$ is a *square* matrix which satisfies $Q^tQ = I$. Then $Q^t = Q^{-1}$ and the matrix is called **orthogonal**. A matrix is orthogonal if its columns (rows) form an orthonormal basis.

Ex. Let find QR-decomposition of the matix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}.$$

First, apply the Gram-Schmidt orthogonalization to the columns of A,

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \; v_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \Longrightarrow$$

$$e_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \; e_2 = c(v_2 - \frac{\langle v_2, e_1\rangle}{\langle e_1, e_1\rangle} e_1) = \frac{1}{2\sqrt{5}} \begin{bmatrix} -3 \\ -1 \\ 1 \\ 3 \end{bmatrix}.$$

Thus

$$Q = [e_1 \; e_2] = \begin{bmatrix} \frac{1}{2} & \frac{-3}{2\sqrt{5}} \\ \frac{1}{2} & \frac{-1}{2\sqrt{5}} \\ \frac{1}{2} & \frac{1}{2\sqrt{5}} \\ \frac{1}{2} & \frac{3}{2\sqrt{5}} \end{bmatrix}, \quad R = \begin{bmatrix} \langle v_1, e_1\rangle & \langle v_2, e_1\rangle \\ 0 & \langle v_2, e_2\rangle \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 0 & \sqrt{5} \end{bmatrix}.$$

Least square solution
Let $A \in M_{m\times n}(\mathbb{R})$ and $b \in \mathbb{R}^m$ are such that the system $Ax = b$ has no solution. Then the **least squares solution** is $\hat{x}$ to $Ax = b$ is defined as a vector in $\mathbb{R}^n$ for which $|A\hat{x} - b|$ is least possible. Thus $A\hat{x}$ is the closest point to b in $\text{ran}(A)$. Hence $A\hat{x}$ is the orthogonal projection of b onto $\text{ran}(A)$.

Let further $A = (v_1, ..., v_n)$ and $\{e_1, ..., e_n\}$ be the orthonormal basis obtained from $\{v_1, ..., v_n\}$ by the Gram-Schmidt algorithm. Then

$$A\hat{x} = \sum_{j=1}^{n} \langle b, e_j \rangle e_j = QQ^t b.$$

Then the QR-decomposition implies the normal equation

$$A^t A\hat{x} = (QR)^t QQ^t b = R^t (Q^t Q) Q^t b = R^t Q^t b = A^t b.$$

Moreover, instead of considering the normal system, one can multiply $A\hat{x} = QQ^t b$ by Q^t and obtain

$$R\hat{x} = Q^t b,$$

since $Q^t A = Q^t QR = R$. The last system can be solved by back substitution since R is upper triangular.

Finally, the orthogonal projection of b onto $\text{ran}(A)$ equals $A\hat{x}$. If A is of rank n then R is invertible and

$$A\hat{x} = A(R^{-1}Q^t)b = QQ^t b.$$

The matrix of orthogonal projection onto $\text{ran}(A)$ is QQ^t.

Ex. In the previous example, the matrix of orthogonal projection onto $\text{ran}(A)$ is

$$QQ^t = \begin{bmatrix} \frac{1}{2} & \frac{-3}{2\sqrt{5}} \\ \frac{1}{2} & \frac{-1}{2\sqrt{5}} \\ \frac{1}{2} & \frac{1}{2\sqrt{5}} \\ \frac{1}{2} & \frac{3}{2\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{-3}{2\sqrt{5}} & \frac{-1}{2\sqrt{5}} & \frac{1}{2\sqrt{5}} & \frac{3}{2\sqrt{5}} \end{bmatrix} = \frac{1}{20} \begin{bmatrix} 19 & 13 & 7 & 1 \\ 13 & 11 & 9 & 7 \\ 7 & 9 & 11 & 13 \\ 1 & 7 & 13 & 19 \end{bmatrix}.$$

Chapter 4

Linear transformations

4.1 Linear transformations and matrices

Let X and Y be vector spaces and $T : X \to Y$ a mapping between them.

Definition and examples of linear transformations

We say that T is a **linear transformation** (or just **linear**) if it preserves the linear structure of a vector space:

$$T \text{ linear} \quad \overset{\text{def}}{\iff} \quad T(\lambda x + \mu y) = \lambda Tx + \mu Ty, \qquad x, y \in X,\ \mu, \lambda \in \mathbb{R} \text{ (or } \mathbb{C}).$$

Ex.

- Any matrix $A \in M_{m\times n}(\mathbb{R})$ defines a linear transformation $\mathbb{R}^n \to \mathbb{R}^m$:

$$\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{x} \mapsto \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{Ax}.$$

- The integral operator defined by $Tf(t) := \int_0^t f(s)\,ds$ is a linear transformation on $C(I, \mathbb{R})$:

$$T\colon C(I,\mathbb{R}) \to C(I,\mathbb{R}), \quad Tf = \left[t \mapsto \int_0^t f(s)\,ds\right].$$

- A slight modification,

$$Tf := \int_0^1 f(s)\,ds,$$

yields a linear transformation $C(I, \mathbb{R}) \to \mathbb{R}$ (given that $[0, 1] \subset I$).[1]

[1] Such an operator is called a **linear functional**.

- For any polynomial $p \in P_k(\mathbb{R})$, the differential operator $p(D) := \sum_{j=0}^{k} a_j D^j$ is a linear transformation:

$$p(D)\colon C^k(I,\mathbb{R}) \to C(I,\mathbb{R}), \qquad p(D)f = \sum_{j=0}^{k} a_j f^{(j)}.$$

  Here, $D = \frac{d}{dx}$ is the standard differentiation operator.

- The shift operator $T\colon (x_1, x_2, \ldots) \mapsto (0, x_1, x_2, \ldots)$, is a linear transformation $l_p \to l_p$, for any p, $1 \le p \le \infty$:

$$T(\lambda x + \mu y) = (0, \lambda x_1 + \mu y_1, \ldots) = \lambda(0, x_1, \ldots) + \mu(0, y_1, \ldots) = \lambda Tx + \mu Ty.$$

  Note that $\|Tx\|_{l_p} = \|x\|_{l_p}$ guarantees that $\operatorname{ran}(T) \subset l_p$.

Linear transformations between finite-dimensional spaces

℘ **A linear transformation is determined by its action on any basis**

Let X be a finite-dimensional[1] vector space with basis $\{e_1, \ldots, e_n\}$. For any values $y_1, \ldots, y_n \in Y$ there exists exactly one linear transformation $T \in L(X,Y)$ such that

$$Te_j = y_j, \qquad j = 1, \ldots, n.$$

Proof

Any $x \in X$ has a unique representation $x = \sum_{j=1}^{n} x_j e_j$. Define T through

$$Tx = \sum_{j=1}^{n} x_j y_j.$$

Then $Te_j = y_j$, and T is linear since it acts as multiplication with a $1 \times n$ matrix (a dot product with the vector $(y_1, \ldots, y_n)$). Moreover, if $S \in L(X,Y)$ also satisfies $Se_j = y_j$, then

$$Sx = S\big(\sum_{j=1}^{n} x_j e_j\big) = \sum_{j=1}^{n} x_j Se_j = \sum_{j=1}^{n} x_j y_j = Tx, \qquad \text{for all } x \in X,$$

so that $S = T$ in $L(X,Y)$.

Ex. The columns A_j of an $m \times n$-matrix A are images of the on the standard basis $\{e_j\}_{j=1}^{n}$ vectors under the action of A,

$$Ae_j = A_j, \qquad j = 1, \ldots, n.$$

Here A_j plays the role of y_j in the above theorem. The columns of A can be chosen arbitrary and they determine the matrix uniquely.

℘ **Linear transformations between finite-dimensional vector spaces correspond to matrices**

Let X, Y be real vector spaces of dimension n and m, respectively. Then there is a bijection between $L(X,Y)$ and $M_{m\times n}(\mathbb{R})$.

[1] For infinite-dimensional Banach spaces one needs the additional concept of **boundedness** (continuity) of a linear transformation to state a similar result, which then says that the transformation is determined by Te_j (but we cannot choose $Te_j = y_j$ arbitrarily).

N.b. The corresponding statement holds for complex vector spaces X, Y, with $M_{m\times n}(\mathbb{C})$ also complex-valued.

Proof
Since $X \cong \mathbb{R}^n$ and $Y \cong \mathbb{R}^m$ it suffices to prove the statement for these choices of X and Y. Let $\{e_j\}_{j=1}^n$ be the standard basis for $\mathbb{R}^n$. Then

$$T\colon \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \mapsto \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

is a linear transformation $\mathbb{R}^n \to \mathbb{R}^m$ satisfying

$$Te_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}.$$

According to the above proposition, there is exactly one such $T \in L(\mathbb{R}^n, \mathbb{R}^m)$. Since we can choose the columns of $A = (a_{ij})_{ij}$ to be any elements in $\mathbb{R}^m$, we get all possible $T \in L(\mathbb{R}^n, \mathbb{R}^m)$ in this way.

Ex.

- The linear transformation $T\colon (x_1, x_2) \mapsto (-x_2, x_1)$ on $\mathbb{R}^2$ is realized by a rotation matrix A:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}.$$

- More generally, the rotation on θ radians counterclockwise is given by[1]

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix},$$

  the preceding example is attained for $\theta = \pi/2$.
- The differential operator $\frac{d}{dx}$ is a linear operator on $P_2(\mathbb{R})$. Since $P_2(\mathbb{R}) \cong \mathbb{R}^3$ via the vector space isomorphism

$$\sum_{j=0}^{2} a_j x^j \overset{\varphi}{\mapsto} (a_0, a_1, a_2),$$

  we see that the derivation $\frac{d}{dx}\left(a_0 + a_1 x + a_2 x^2\right) = a_1 + 2a_2 x + 0x^2$ is expressed by a matrix

$$\frac{d}{dx}\colon \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ 2a_2 \\ 0 \end{bmatrix}.$$

The set of linear transformations as a vector space

The set of linear transformations $X \to Y$ is denoted by $L(X, Y)$:

$$L(X, Y) \overset{\text{def.}}{=} \{T\colon X \to Y \text{ linear}\}$$

If $X = Y$, we may abbreviate $L(X, X)$ by $L(X)$.

[1] The determinant of the matrix is 1, so it is invertible, regardless of the value of θ.

℘ **L(X,Y) is a vector space**
If, for all $S, T \in L(X, Y)$, we define

$$(T + S)(x) := Tx + Sx \quad \text{and} \quad (\lambda T)x := \lambda(Tx),$$

for all $x \in X$ and $\lambda \in \mathbb{R}$ (or $\mathbb{C}$), it is easily checked that $L(X, Y)$ becomes a vector space. In particular, $\mu T + \lambda S \in L(X, Y)$ for any $S, T \in L(X, Y)$.

Ex. The set of $m \times n$-matrices $M_{m\times n}(\mathbb{R})$ forms a real vector space. The bijection between $L(\mathbb{R}^n, \mathbb{R}^m)$ and $M_{m\times n}(\mathbb{R})$ described above is an isomorphism of vector spaces

$$M_{m\times n}(\mathbb{R}) \cong L(\mathbb{R}^n, \mathbb{R}^m).$$

Composition of linear transformations and matrix multiplication

Let $T : X \to Y$ and $S : Y \to Z$ be mappings between vector spaces.

℘ **Composition of linear transformations is a linear transformation**
If T and S are linear then $S \circ T : X \to Z$ is also linear.

Proof
For any $x, y \in X$ and $\lambda, \mu \in \mathbb{R}$ (or $\mathbb{C}$)

$$\begin{aligned} S \circ T(\lambda x + \mu y) &= S(T(\lambda x + \mu y)) = S(\lambda T(x) + \mu T(y)) \\ &= \lambda S(T(x)) + \mu S(T(y)) = \lambda S \circ T(x) + \mu S \circ T(y). \end{aligned}$$

Matrix multiplication
Let $A = \{a_{ij}\}_{ij}$ be an $m \times n$ matrix and $B = \{b_{kl}\}_{kl}$ be an $n \times p$ matrix. Then the **product** of A and B is the $m \times p$ matrix $C = \{c_{ij}\}_{ij}$ such that

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ and $S : \mathbb{R}^p \to \mathbb{R}^n$ be linear transformations with matrices A in B in standard bases. Then the matrix of $S \circ T$ in the standard bases is AB.

Ex. Composition of two rotations is a new rotation, this yields

$$\begin{bmatrix} \cos(\theta + \varphi) & -\sin(\theta + \varphi) \\ \sin(\theta + \varphi) & \cos(\theta + \varphi) \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{bmatrix}$$

and trigonometric identities

$$\begin{aligned} \cos(\theta + \varphi) &= \cos(\theta)\cos(\varphi) - \sin(\theta)\sin(\varphi), \\ \sin(\theta + \varphi) &= \sin(\theta)\cos(\varphi) + \cos(\theta)\sin(\varphi). \end{aligned}$$

4.2 Gaussian elimination and LU-factorization

Linear systems

Any linear system of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \quad & \quad \vdots \quad \quad \vdots \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

where $a_{ij}, b_i \in \mathbb{C}$ for $i = 1, \ldots, m$, $j = 1, \ldots, n$, and $x_1, \ldots, x_n$ are unknowns, can be written in matrix form:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

Finding solutions $(x_1, \ldots, x_n$ of the linear system of equations is then equivalent to the following question: given a matrix $A \in M_{m\times n}(\mathbb{C})$ and a vector $b \in \mathbb{C}^m$, is there a vector $x \in \mathbb{C}^n$ such that $Ax = b$?[1]

The answer depends on A and, for some A, also on b.

Ex.

- $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \iff \begin{matrix} x_1 = 1 \\ 2x_1 = 1 \end{matrix}$ has no solution.
- $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \iff \begin{matrix} x_1 = 1 \\ 2x_1 = 2 \end{matrix}$ has infinitely many solutions: $x_1 = 1, x_2 \in \mathbb{R}$.
- $\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \iff \begin{matrix} x_1 + x_2 = b_1 \\ 2x_1 \quad\quad = b_2 \end{matrix}$ has a unique solution for any b_1, b_2 : $x_1 = \frac{b_2}{2}, x_2 = b_1 - \frac{b_2}{2}$.

Gaussian elimination and the row echelon form of a matrix

A matrix is in **row echelon form** if i) the left-most non-zero entry of each row (**pivot**) is strictly to the right of the left-most non-zero entry of any row above, and ii) all-zero rows are at the bottom of the matrix:

$$\begin{bmatrix} 1 & \cdots & & & \\ 0 & 2 & \cdots & & \\ 0 & 0 & 0 & 7 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \end{bmatrix}$$

[1] When A and b are real, one looks for x real.

℘ **Any linear system can be brought into row echelon form**

Proof

If $A = 0$ is the zero matrix, we are done.

Else, assume for simplicity that $a_{11} \neq 0$ (If not rearrange the rows, or relabel the x_j's, or both). To each row $(a_{i1}, \dots, a_{in})$ below the first, add $-\frac{a_{i1}}{a_{11}} \times$ the first row:

$$\begin{array}{rr} a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n = b_1 & -\dfrac{a_{i1}}{a_{11}} \\ a_{i1}x_1 + a_{i2}x_2 + \cdots a_{in}x_n = b_2 & \\ \hline 0 + \left(a_{i2} - \dfrac{a_{i1}}{a_{11}}a_{12}\right)x_2 + \cdots \left(a_{in} - \dfrac{a_{i1}}{a_{11}}a_{1n}\right)x_n = b_2 - \dfrac{a_{i1}}{a_{11}}b_1 & \end{array}$$

Then $a_{i1} = 0$ for all $i = 2, \dots, m$.

Now, either $a_{ij} = 0$ for all $i, j \geq 2$, or we can restart this procedure (looking at the matrix for indices $i, j \geq 2$). Since there are finitely many rows this procedure must eventually terminate, yielding a matrix $\tilde{A} = (\tilde{a}_{ij})_{ij}$ in row echelon form.

This algorithm is called **Gaussian elimination**.

A neat trick is the following: write $Ax = b$ as $Ax = Ib$:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & 0 & \ddots & \dots \\ 0 & & \cdots & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \qquad I \in M_{m\times m}.$$

Gaussian elimination affects only the matrices A and I, not the vectors x and b. We therefore introduce the $m \times (n + m)$ **augmented matrix**

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & 1 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & 0 & \ddots & \dots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & 0 & & \cdots & 1 \end{bmatrix}.$$

The linear operations applied to A in Gaussian elimination are 'stored' in the augmented matrix.

LU-decompositions

The operations on the rows used in the Gauss elimination are additions of a multiple of one row to another and permutations of rows. First assume that only operations of the first kind were applied. Each such operation, add the jth row multiplied by c to kth row, corresponds to multiplication of A by a matrix E from the left, where E has one's on the main diagonal, c on the intersection of kth row and jth column and zeros otherwise. The Gauss elimination applied to the augmented matrix yields

$$[A|I] \ \rightarrow \ [U|E_s...E_1],$$

where $U = E_s...E_1A$ is a row echelon matrix and each E_i is of the form

$$E_i = \begin{bmatrix} 1 & ... & 0 & ... & 0 & ... & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ 0 & ... & 1 & ... & 0 & ... & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ 0 & ... & c & ... & 1 & ... & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ 0 & ... & 0 & ... & 0 & ... & 1 \end{bmatrix}.$$

It is easy to see that E_i^{-1} corresponds to the inverse row operation and is given by

$$E_i^{-1} = \begin{bmatrix} 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ 0 & \dots & 1 & \dots & 0 & \dots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ 0 & \dots & -c & \dots & 1 & \dots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{bmatrix}.$$

The row operation in Gauss elimination are always done in such way that a multiple of one row is added to another row which is below the first one. Then all E_i and E_i^{-1} are lower triangular matrices with ones on the main diagonal, their product is also a lower triangular matrix with ones on the main diagonal and we get $\tilde{L}A = U$, where $\tilde{L} = E_s...E_1$. Let $L = \tilde{L}^{-1} = E_1^{-1}...E_s^{-1}$, then $A = LU$, where L is lower triangular and U is an echelon form matrix. This is the **LU-decomposition** of the matrix A (it does not always exist).[1]

N.b. If the rows of A are not in correct order (for example, if $a_{11} = 0$), they can be rearranged by applying a **permutation matrix**[2] P:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \text{row 1} \\ \text{row 2} \\ \text{row 3} \end{bmatrix} = \begin{bmatrix} \text{row 2} \\ \text{row 3} \\ \text{row 1} \end{bmatrix}.$$

This is known as an **LUP-factorization**: $PA = LU$. For square matrices, an LUP-factorization always exists (but it is not necessarily unique).

Ex. Solve

$$\underbrace{\begin{bmatrix} 1 & 3 & 1 \\ 2 & 2 & 0 \\ 2 & 2 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}}_{b} \iff \begin{bmatrix} 1 & 3 & 1 \\ 2 & 2 & 0 \\ 2 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}.$$

Now, performing the same row operations on the whole augmented matrix,

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ 2 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \iff \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & -4 & -2 & -2 & 1 & 0 \\ 0 & -4 & -3 & -2 & 0 & 1 \end{array}\right] \iff \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & -4 & -2 & -2 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 \end{array}\right],$$

we find the row echelon form

$$\underbrace{\begin{bmatrix} 1 & 3 & 1 \\ 0 & -4 & -2 \\ 0 & 0 & -1 \end{bmatrix}}_{U\text{: upper triangular}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}_{\tilde{L}\text{: lower triangular}} \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}.$$

Note that $\tilde{L}$ describes the (linear) transformation applied to A to obtain U: we have $\tilde{L}A = U$.

[1] When it exists, it is unique if one requires the diagonal elements of L to be all ones.
[2] A permutation matrix has (a permutation of) the standard basis $\{e_j\}_j$ as rows.

Ex. There are two ways to find L.

- Gaussian elimination for $\tilde{L}$ yields

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \iff \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \iff \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 & 1 \end{array}\right],$$

  meaning that

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}}_{L=\tilde{L}^{-1}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}_{\tilde{L}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

- Tracing the row operations of the Gauss elimination, we see that

$$\tilde{L} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

  And

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$$

Hence, the LU-factorization of A is

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & 2 & 0 \\ 2 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 0 & -4 & -2 \\ 0 & 0 & -1 \end{bmatrix}.$$

4.3 Basis transformations

Change-of-basis matrix

Let $e = \{e_1, \ldots, e_n\}$ and $f = \{f_1, \ldots, f_n\}$ be two bases for a finite-dimensional real vector space X. Pick any element $x \in X$. Then

$$x = \sum_{j=1}^{n} x_j e_j \quad \text{and} \quad x = \sum_{j=1}^{n} y_j f_j,$$

it has coordinates $x_e = (x_1, \ldots, x_n)$ in the basis e and coordinates $x_f = (y_1, \ldots, y_n)$ in the basis f. The vectors e_j can be also expressed in the basis f,

$$e_j = \sum_{k=1}^{n} c_{k,j}\, f_k, \quad j = 1, \ldots, n, \qquad \text{and} \qquad x = \sum_{j=1}^{n} x_j \sum_{k=1}^{n} c_{k,j}\, f_k = \sum_{k=1}^{n} \Bigg(\underbrace{\sum_{j=1}^{n} c_{k,j}\, x_j}_{\text{coord. in } f} \Bigg) f_k.$$

The uniqueness of coordinates then implies

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,n} \\ c_{2,1} & c_{2,2} & \cdots & c_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n,1} & c_{n,2} & \cdots & c_{n,n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

or $x_f = Cx_e$. The $n \times n$ matrix $C \in M_{n\times n}(\mathbb{R})$ is called a **change-of-basis matrix**.

Ex.

- Let (e_1, e_2) be the standard basis for $\mathbb{R}^2$ and $f_1 = (\cos(\theta), \sin(\theta))$ and $f_2 = (-\sin(\theta), \cos(\theta))$ be the rotated basis. To find the change-of-basis matrix we write vectors of the standard basis as linear combinations of the elements of the second basis

$$e_1 = \cos(\theta)f_1 - \sin(\theta)f_2, \quad e_2 = \sin(\theta)f_1 + \cos(\theta)f_2.$$

  Then the rotation (by angle $-\theta$) matrix

$$C = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

  is the change-of-basis matrix;

$$\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

  expresses the coordinates of x in basis f, where $x_e = (x_1, x_2)$.
- The change-of-basis matrix from

$$e = \{(1,0,0),(0,1,0),(0,0,1)\} \quad \text{to} \quad f = \{(1,0,0),(1,1,0),(1,1,1)\} \quad \text{in } \mathbb{R}^3$$

$$\begin{aligned} e_1 = \sum_{k=1}^{3} c_{k,1} f_k \quad &\iff \quad \begin{bmatrix} c_{1,1} \\ c_{2,1} \\ c_{3,1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ e_2 = \sum_{k=1}^{3} c_{k,2} f_k \quad &\iff \quad \begin{bmatrix} c_{1,2} \\ c_{2,2} \\ c_{3,2} \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \\ e_3 = \sum_{k=1}^{3} c_{k,3} f_k \quad &\iff \quad \begin{bmatrix} c_{1,3} \\ c_{2,3} \\ c_{3,3} \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \end{aligned}$$

  The change-of-basis matrix is

$$C = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

  In particular,

$$\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \qquad \text{yields that} \qquad (2,0,1)_e = (2-1,1)_f.$$

Change-of-basis matrix as an inverse

If we write the identities from the last example in column form, we get:

$$\begin{bmatrix} e_1 e_2 e_3 \end{bmatrix} = \begin{bmatrix} f_1 f_2 f_3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \iff \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus $I = [f]C$ and $C = [f]^{-1}$, where $[f]$ is the matrix with the basis vectors $f_1, \ldots, f_n$ as column vectors.

℘ **The inverse of a basis matrix is its inverse change-of-basis matrix**
Let $[f] = [f_1, \ldots, f_n] \in M_{n\times n}(\mathbb{C})$ denote a matrix with column basis vectors $f_1, \ldots, f_n \in \mathbb{C}^n$ expressed in the standard basis e. Then

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}_e = \underbrace{\left| \begin{bmatrix} . \\ f_1 \\ . \end{bmatrix} \cdots \begin{bmatrix} . \\ f_n \\ . \end{bmatrix} \right|}_{[f]} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}_f \qquad \text{expresses } (y_1, \ldots, y_n)_f \text{ in the basis } e,$$

and

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}_f = \underbrace{\left| \begin{bmatrix} . \\ f_1 \\ . \end{bmatrix} \cdots \begin{bmatrix} . \\ f_n \\ . \end{bmatrix} \right|^{-1}}_{[f]^{-1}} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}_e \qquad \text{expresses } (x_1, \ldots, x_n)_e \text{ in the basis } f.$$

℘ **Any basis in a finite-dimensional vector space corresponds to an invertible matrix**
Let e be a basis for an n-dimensional vector space X. Then there is a bijection between bases for X and invertible $n \times n$-matrices, $f \mapsto [f]$.

Proof
The coordinates of vectors in e define an isomorphism $X \cong \mathbb{K}^n$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Each basis f defines an invertible matrix $[f]$. Now let A be an invertible $n \times n$ matrix over $\mathbb{K}$ and let $A = [A_1, \ldots, A_n]$, where $A_1, ...A_n \in \mathbb{K}^n$. Since A is invertible, the vectors $A_1, ..A_n$ are linear independent and thus form a basis for $\mathbb{K}^n$. Let $f_j \in X$ have coordinates A_j in the basis e. Then $f_1, ..., f_n$ is a basis for X and $[f] = A$.

Representing linear transformations in different bases
Let $e = \{e_1, \ldots, e_n\}$ (first basis) and $f = \{f_1, \ldots, f_n\}$ (new basis) be two bases for X, and $[f]$ the matrix with $[f_j]$, $j = 1, \ldots, n$, as column vectors (expressed in the first basis e). Then

$$x_e = [f]x_f \quad \text{and} \quad x_f = [f]^{-1}x_e.$$

Hence, if

$$T \in L(X): \quad T \text{ is realized by } A_e \in M_{n\times n}(\mathbb{R}) \text{ in the basis } e,$$

what is its realization A_f in the basis f? The following identities hold

$$(T(x))_e = A_e x_e \iff (T(x))_f = [f]^{-1}(T(x))_e = [f]^{-1}A_e x_e = [f]^{-1}A_e[f]x_f.$$

Thus

$$A_f = [f]^{-1}A_e[f]$$

is the realization of T in the basis f.

Ex. How do we express the rotation

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_e \mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_e = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}_e$$

in the basis $f = \{(1,1), (-1,0)\}$? Since

$$[f] = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad [f]^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix},$$

we have

$$A_f = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}}_{[f]^{-1}} \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{A_e} \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}}_{[f]} = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}.$$

Check: $(x_1, x_2)_e \overset{[f]^{-1}}{\mapsto} (x_2, x_2 - x_1)_f \overset{A_f}{\mapsto} (x_1, x_1 + x_2)_f \overset{[f]}{\mapsto} (-x_2, x_1)_e$ describes the correct transformation.

Similar matrices

Let $A, B \in M_{n\times n}(\mathbb{F})$, they are called **similar** if there exists an invertible matrix Q such that $B = Q^{-1}AQ$.

It is not difficult to check that the definition is symmetric, $B = Q^{-1}AQ$ implies $A = QBQ^{-1}$, if A is similar to B and B is similar to C then A is similar to C. Thus the set of all $n \times n$ matrices (over $\mathbb{K}$) is partitioned into classes of similar matrices. The change of basis formula means that two matrices are similar if and only if they are matrices of the same linear transformation on $\mathbb{K}^n$ in different bases.

Ex. The example above shows that the following matrices are similar

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}.$$

4.4 Kernels and ranges of linear transformations

Let X and Y be vector spaces (both real, or both complex), and $T\colon X \to Y$ a mapping between them.

Kernels and ranks

Notions of the kernel and range discussed for matrices can be generalized to arbitrary linear transformations between vector spaces.

Let $T \in L(X, Y)$. The set of vectors for which T vanishes is called the **kernel of** T,

$$\ker(T) \overset{\text{def.}}{=} \{x \in X : Tx = \mathbf{0} \text{ in } Y\}.$$

The range ot T is as usual

$$\operatorname{ran}(T) \overset{\text{def.}}{=} \{y \in Y : \exists x \in X; y = Tx\}.$$

When the dimension of $\operatorname{ran}(T)$ is finite it is called the **rank of** T, $\operatorname{rank}(T)$ and T is said to be an operator of **final rank**.

℘ **The kernel and range of a linear transformation are vector spaces**
Let $T \in L(X, Y)$. Then $\ker(T) \subset X$ is a linear subspace of X, and $\operatorname{ran}(T) \subset Y$ is a linear subspace of Y.

Proof
For the kernel of T: If $x_1, x_2 \in \ker(T)$, then

$$T(\lambda x_1 + \mu x_2) = \lambda T x_1 + \mu T x_2 = \lambda \mathbf{0} + \mu \mathbf{0} = \mathbf{0}.$$

This shows that $\ker(T)$ is a subspace of X. (Note, in particular, that the zero element of X is always in $\ker(T)$.)

For the range of T: If $y_1, y_2 \in \operatorname{ran}(T)$, then there exists $x_1, x_2 \in X$ such that

$$Tx_1 = y_1, \qquad Tx_2 = y_2.$$

By the linearity of T, for any scalars λ, μ,

$$\lambda y_1 + \mu y_2 = \lambda T x_1 + \mu T x_2 = T(\lambda x_1 + \mu x_2) \in \operatorname{ran}(T),$$

where $\mu x_1 + \lambda x_2 \in X$, by the properties of a vector space. Thus $\lambda y_1 + \mu y_2 \in \operatorname{ran}(T)$.

Ex.

- The kernel of $T \in L(\mathbb{R}^2)$: $(x_1, x_2) \mapsto (-x_2, x_1)$ is the trivial subspace $\{(0,0)\} \subset \mathbb{R}^2$. Since $\operatorname{ran}(T) = \mathbb{R}^2$, we have $\operatorname{rank}(T) = 2$.
- The differential operator $\frac{d}{dx}$ is a linear operator $C^1(\mathbb{R}) \to C(\mathbb{R})$[1]. As we know,

  $$\ker\Big(\frac{d}{dx}\Big) = \{f \in C^1(\mathbb{R})\colon f(x) \equiv c \text{ for some } c \in \mathbb{R}\},$$

  so that $\ker(\frac{d}{dx}) \cong \mathbb{R}$ is a one-dimensional subspace of $C^1(\mathbb{R})$. Since

  $$\frac{d}{dx}\int_0^x f(t)\,dt = f(x) \qquad \text{for any } f \in C(\mathbb{R}),$$

  we have $\operatorname{ran}(\frac{d}{dx}) = C(\mathbb{R})$ and $\operatorname{rank}(\frac{d}{dx}) = \infty$.
- *The domain of definition matters*: considered as an operator on $P_n(\mathbb{R})$ the differential operator $\frac{d}{dx}\colon P_n(\mathbb{R}) \to P_n(\mathbb{R})$ still has a one-dimensional kernel (the space of constant polynomials, $P_0(\mathbb{R})$), but its range is now finite-dimensional:

  $$\operatorname{ran}\Big(\frac{d}{dx}\Big) = P_{n-1}(\mathbb{R}) \cong \mathbb{R}^n.$$

℘ **A linear transformation is injective if and only if its kernel is trivial**
Let $T \in L(X, Y)$. Then T injective $\iff$ $\ker(T) = \{\mathbf{0}\}$.

Proof

$$\begin{aligned} T \text{ injective} &\iff \big[Tx = Ty \Longrightarrow x = y\big] \iff \big[T(x-y) = 0 \Longrightarrow x - y = 0\big] \\ &\iff \big[Tz = 0 \Longrightarrow z = 0\big] \iff \ker(T) = \{0\}. \end{aligned}$$

[1] Here we use the convention that $C^k(\mathbb{R}) = C^k(\mathbb{R}, \mathbb{R})$, just as one may write $L(X) = L(X, X)$ for linear transformations on a space X.

Ex. A matrix $A \in M_{m\times n}(\mathbb{R})$ describes a linear transformation $\mathbb{R}^n \to \mathbb{R}^m$. This transformation is injective if zero (the zero element in $\mathbb{R}^n$) is the only solution of the corresponding linear homogeneous system:

$$\begin{array}{cccc} a_{11}x_1 & +\ldots+ & a_{1n}x_n & = 0 \\ \vdots & & \vdots & \\ a_{m1}x_1 & +\ldots+ & a_{mn}x_n & = 0 \end{array} \quad \Longrightarrow \quad (x_1, \ldots, x_n) = (0, \ldots, 0).$$

The inverse operator theorem

Suppose that $T \in L(X, Y)$ is injective then there exists a unique linear transformation

$$S : \operatorname{ran}(T) \to X$$

such that $S \circ T$ is the identity operator on X. Further $T(S(y)) = y$ for any $y \in \operatorname{ran}(Y)$, S is called the **left inverse of** T.

Proof

Clearly for any $y \in \operatorname{ran}(T)$ there exists unique $x \in X$ such that $T(x) = y$, define $S(y) = x$. The linearity of S follows from the linearity of T. Suppose that $T(x_1) = y_1$ and $T(x_2) = y_2$ then $T(\lambda x_1 + \mu x_2) = \lambda y_1 + \mu y_2$ and

$$S(\lambda y_1 + \mu y_2) = \lambda y_1 + \mu \lambda y_2).$$

If $S(y) = x$ then $T(S(y)) = y$ by the definition of S.

Ex.

- Suppose that $T \in L(\mathbb{R}^n, \mathbb{R}^m)$ is injective, i.e., $\operatorname{rank}(T) = n$. Then

  $$\exists S \in L(\mathbb{R}^m, \mathbb{R}^n) \text{ such that } S \circ T : \mathbb{R}^n \to \mathbb{R}^n \text{ is the identity map.}$$

  The left inverse is first defined only on ran(T) that is a subspace of $\mathbb{R}^m$. It can be extended to a linear operator from $\mathbb{R}^m$ to $\mathbb{R}^n$. On the language of matrices, if A is an $m \times n$ matrix of rank n then there exists an $n \times m$ matrix B such that $BA = I_n$.
- Let $T : l^2 \to l^2$ be the right shift and $S : l^2 \to l^2$ be the left shift,

  $$T : (x_1, x_2, ...) \mapsto (0, x_1, x_2, ...), \quad S : (x_1, x_2, ...) \mapsto (x_2, ...,).$$

  Then T is injective and $S \circ T$ is the identity map. Note that S is not injective and has no left inverse, $T \circ S(x) = x$ only on the range of T.

The rank–nullity theorem and its consequences

℘ **The rank–nullity theorem**

Let $T \in L(V, W)$, where V and W are vector spaces and V is finite dimensional, then

$$\dim \ker(T) + \dim \operatorname{ran}(T) = \dim V.$$

Proof

Pick a basis $e = \{e_1, \ldots, e_k\}$ for $\ker(T)$. If $k = n := \dim V$ then $\ker(T) = V$ we are done, since then

$$\operatorname{ran}(T) = \{Tx \colon x \in V\} = \{0\},$$

so that $\dim \ker(T) + \dim \operatorname{ran}(T) = n$.

Hence, assume that $k < n$ and extend e to a basis $\{e_1, \ldots, e_k, f_1, \ldots, f_m\}$ for $\mathbb{R}^n$.

This can be done in the following way: pick $f_1 \notin \text{span}\{e_1, \ldots, e_k\}$. Then $\{e_1, \ldots, e_n, f_1\}$ is linearly independent. If $\text{span}\{e_1, \ldots, e_k, f_1\} = v$ we stop. Else, pick $f_2 \notin \text{span}\{e_1, \ldots, e_k, f_1\}$. Since $l > n$ vectors are always linearly dependent in v, this process stops when $k + m = n$ (it cannot stop before, since then v would be spanned by a set of less than n vectors which is impossible; see the definition of vector space dimension).

We now prove that $Tf = \{Tf_1, \ldots Tf_m\}$ is a basis for $\text{ran}(T)$.

Tf is linearly independent:

$$\sum_{j=1}^{m} a_j T f_j = 0 \iff T\big(\sum_{j=1}^{m} a_j f_j\big) = 0 \iff \sum_{j=1}^{m} a_j f_j \in \ker(T) \iff a_j = 0 \,\forall j = 1, \ldots, m,$$

since T is linear, and since, by the construction of f, no non-zero linear combination of elements f_j is in $\ker(T)$.

Furthermore, Tf spans $\text{ran}(T)$:

$$\begin{aligned}\text{ran(T)} = \{Tx \colon x \in \mathbb{R}^n\} &= \Big\{T\big(\sum_{j=1}^{k} a_j e_j + \sum_{j=1}^{m} b_j f_j\big) \colon a_j, b_j \in \mathbb{R}\Big\} \\ &= \Big\{T\big(\sum_{j=1}^{k} a_j e_j\big) + T\big(\sum_{j=1}^{m} b_j f_j\big) \colon a_j, b_j \in \mathbb{R}\Big\} = \Big\{\sum_{j=1}^{m} b_j T f_j \colon b_j \in \mathbb{R}\Big\},\end{aligned}$$

since T is linear and $e \subset \ker(T)$.

Hence, $\{Tf_1, \ldots, Tf_m\}$ is a basis for $\text{ran}(T)$, and

$$\dim \ker(T) + \dim \text{ran}(T) = k + m = n.$$

Corollary: the rank of A and A^t
In the previous chapter we showed that for any $A \in M_{m\times n}(\mathbb{R})$

$$n = \dim(\ker(A)) + \dim(\text{ran}(A^t)).$$

Applying the rank-nullity theorem for the operator defined by A we obtain

$$\dim(\text{ran}(A^t)) = \dim(\text{ran}(A)).$$

Thus matrices A and A^t have the same rank.

℘ **For finite-dimensional linear transformations, injective means surjective**
Let $T \in L(\mathbb{R}^n)$ be a linear transformation $\mathbb{R}^n \to \mathbb{R}^n$. Then the following are equivalent:

(i) T is injective $\quad (\ker(T) = \{\mathbf{0}\})$

(ii) T is surjective $\quad (\text{ran}(T) = \mathbb{R}^n)$

(iii) $T \colon \mathbb{R}^n \to \mathbb{R}^n$ is invertible

(iv) The matrix representation A of T (in any given basis) is invertible.

(v) For any $b \in \mathbb{R}^n$ the system $Ax = b$ has a unique solution x.

Proof

(i) $\iff$ (ii): When $m = n$, the rank–nullity theorem says that $\operatorname{ran}(T) = \mathbb{R}^n$ (so that T is surjective) exactly when $\ker(T) = \{\mathbf{0}\}$ (so that T is injective).

(i,ii) $\iff$ (iii): A function is bijective exactly if it is both invertible and surjective.

(iii) $\iff$ (iv): Given any basis for $\mathbb{R}^n$, T has a unique matrix representation A (defined by its action on the basis vectors). If the inverse matrix A^{-1} exists, then there exists a corresponding linear transformation S such that $ST = ST = \mathrm{id}$ (since $A^{-1}A = AA^{-1} = I$, and the identity map $\mathrm{id}\colon \mathbb{R}^n \to \mathbb{R}^n$ has the identity matrix I as representation in all bases). Thus $S = T^{-1}$ is the inverse of T. If, on the other hand, T^{-1} exists, it must by the same argument have a matrix representation B such that $AB = BA = I$. Hence, $A^{-1} = B$ exists.

(iv) $\iff$ (v): If A is invertible it is immediate that $x = A^{-1}b$ is the unique solution. If, on the other hand, $Ax = b$, has a unique solution x for any b, we construct a matrix B by taking as its columns x_j such that $Ax_j = e_j$, where $\{e_1, \dots, e_n\}$ is the standard basis. This guarantees that $B = A^{-1}$ is the inverse matrix of A. (A less constructive argument would be to note that $Ax = b$ is uniquely solvable for all $b \in \mathbb{R}^n$ exactly if T is invertible.)

℘ **Summary on linear equations (the Fredholm alternative)**

Let $A \in M_{m\times n}(\mathbb{R})$ be the realization of a linear transformation $\mathbb{R}^n \to \mathbb{R}^m$, and consider the linear equation

$$Ax = b.$$

- Either the equation $Ax = b$ is solvable for any b or $A^t y = 0$ has a non-trivial solution. (It follows from the identity $\mathbb{R}^m = \ker(A^t) \oplus \operatorname{ran}(A)$.)
- The equation $Ax = b$ has a solution if and only if $b \perp \ker(A^t)$.
- $m - \dim\ker(A^t) = n - \dim\ker(A)$.

4.5 Bounded linear transformations

Let X and Y be normed spaces (both real, or both complex), and $T \in L(X, Y)$ a linear mapping between them.

Boundedness

A linear mapping $T\colon X \to Y$ is called **bounded**, $T \in B(X, Y)$, if T maps bounded sets into bounded sets:

$$T \in B(X,Y) \quad \overset{\text{def}}{\iff} \quad \exists C; \quad \|Tx\|_Y \le C\|x\|_X \quad \text{for all } x \in X.$$

Thus, if T is bounded, the number

$$\|T\| \overset{\text{def.}}{=} \sup_{x\neq 0} \frac{\|Tx\|_Y}{\|x\|_X}$$

is finite; it is the **(operator) norm of** T.

N.b. In some sources $B(X, Y)$ is denoted by $L(X, Y)$; to us, $L(X, Y)$ is the space of linear transformations between two (not necessarily normed) vector spaces; if X and Y are normed spaces, then $B(X, Y) \subset L(X, Y)$.

From now on, we will not always write out the indices for the norms; just recall that $x \in X$ and $Lx \in Y$.

Ex.

- The integral $\int_0^t f(s)\,ds$ defines a linear transformation on the space of bounded and continuous functions $f\colon [0,1] \to \mathbb{R}$,

$$T\colon BC([0,1],\mathbb{R}) \to BC([0,1],\mathbb{R}), \qquad (Tf)(t) = \int_0^t f(s)\,ds.$$

  This transformation is bounded, since

$$\|Tf\|_{BC([0,1],\mathbb{R})} = \sup_{t\in[0,1]} \Big| \int_0^t f(s)\,ds \Big| \le \int_0^1 \max_{s\in[0,1]} |f(s)|\,ds = \|f\|_{BC([0,1],\mathbb{R})},$$

  so that $\|T\| \le 1$. In fact, $\|T\| = 1$ (can you see why?).
- If $g \in BC([0,1],\mathbb{R})$, a similar argument yields that

$$T\colon BC([0,1],\mathbb{R}) \to BC([0,1],\mathbb{R}), \qquad (Tf)(t) = \int_0^t f(s)g(s)\,ds$$

  is bounded too, with $\|T\| \le \max_{t\in[0,1]} |g(t)| = \|g\|_{BC([0,1],\mathbb{R})}$ (see if you can make this better).
- By the same argument, with $t = 1$ and $g \in BC([0,1],\mathbb{R})$, the definite integral $\int_0^1 f(s)g(s)\,ds$ defines a **bounded linear functional**[1]

$$T\colon BC([0,1],\mathbb{R}) \to \mathbb{R}, \qquad f \mapsto \int_0^1 f(s)g(s)\,ds.$$

- The derivative $\frac{d}{dx}$ is in general *not* a bounded operator.[1] To see why, consider a sequence of functions like

$$f_n(x) := \sin(nx).$$

  These functions are uniformly bounded, but not their derivatives. This indicates that, to solve differential equations, it is better to reformulate them as integral operators.

Equivalence of norm expressions

For $T \in B(X,Y)$,

$$\|T\| = \sup_{x\neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{\|x\|=1} \|Tx\| = \sup_{\|x\|\le 1} \|Tx\|$$

all describe the least possible bound on C such that $\|Tx\| \le C\|x\|$ for all $x \in X$.

Proof

Since T is linear, and since norms are positively homogeneous, we get

$$\frac{\|Tx\|}{\|x\|} = \Big\| \frac{1}{\|x\|} Tx \Big\| = \Big\| T\big(\frac{x}{\|x\|}\big) \Big\|, \qquad x \neq 0.$$

Note that the mapping $S_\lambda \to S_1$, $x \mapsto \frac{x}{\|x\|}$, is bijective. Thus, if $\lambda > 0$, we have

$$\sup_{\|x\|=\lambda} \|Tx\| = \lambda \sup_{\|x\|=1} \|Tx\|.$$

[1] A functional is a function from a vector space to its field of scalars ($\mathbb{R}$ or $\mathbb{C}$).

[1] Unless we consider it on some special space of functions, as the finite-dimensional space of real polynomials of degree less than n.

By considering all fixed, but different, $\lambda > 0$, we see that

$$\sup_{\|x\|=\lambda>0} \frac{\|Tx\|}{\|x\|} = \sup_{\|x\|=1} \|Tx\|, \qquad \text{so that} \qquad \sup_{x\neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{\|x\|=1} \|Tx\|.$$

Then, consider $\lambda \leq 1$ to see that

$$\sup_{\|x\|\leq 1} \|Tx\| \overset{\lambda\leq 1}{\leq} \sup_{\|x\|=1} \|Tx\| \leq \sup_{\|x\|\leq 1} \|Tx\|,$$

where the last inequality follows from the definition of the supremum. This proves the assertion.

℘ B(X,Y) is a normed space

$$B(X,Y) = \{T \in L(X,Y) \colon T \text{ is bounded}\}$$

is a normed space when equipped with the operator norm $\|\cdot\|$.

Proof

We already know that $L(X,Y)$ is a linear space, so it remains to show that $B(X,Y)$ is a subspace and $\|\cdot\|$ a norm on $B(X,Y)$.

Subspace property. Take $T, S \in B(X,Y)$ and μ, λ scalars. Then

$$\begin{aligned}\sup_{\|x\|\leq 1} \|(\mu T + \lambda S)(x)\|_Y &\leq \sup_{\|x\|\leq 1} (|\mu|\|Tx\|_Y + |\lambda|\|Sx\|_Y) \\ &\leq |\mu| \sup_{\|x\|\leq 1} \|Tx\|_Y + |\lambda| \sup_{\|x\|\leq 1} \|Sx\|_Y = |\mu|\|T\| + |\lambda|\|S\|\end{aligned}$$

is finite by choice of S, T. Thus $\mu T + \lambda S$ is bounded if T and S are bounded, so that $B(X,Y)$ is a subspace of $L(X,Y)$.

Norm properties. These are consequences of that the operator norm $\|\cdot\|$ is defined using (primarily) the norm of Y.

Positive definiteness:

$$\|T\| = 0 \quad \Longleftrightarrow \quad \|Tx\|_Y = 0 \quad \forall x \in X \quad \Longleftrightarrow \quad T \equiv 0 \quad \text{in } L(X,Y).$$

Positive homogeneity:

$$\|\lambda T\| = \sup_{\|x\|_X \leq 1} \|\lambda T x\|_Y = |\lambda| \sup_{\|x\|_X \leq 1} \|Tx\|_Y = |\lambda|\|T\|.$$

Triangle inequality:

$$\begin{aligned}\|T + S\| = \sup_{\|x\|_X \leq 1} \|(T+S)x\|_Y &\leq \sup_{\|x\|_X \leq 1} \big(\|Tx\|_Y + \|Sx\|_Y\big) \\ &\leq \sup_{\|x\|_X \leq 1} \|Tx\|_Y + \sup_{\|x\|_X \leq 1} \|Sx\|_Y = \|T\| + \|S\|.\end{aligned}$$

Ex.

- As we shall see, $B(\mathbb{R}^n, \mathbb{R}^m) = L(\mathbb{R}^n, \mathbb{R}^m)$ (as sets and linear spaces): given bases for $\mathbb{R}^n, \mathbb{R}^m$ there is a bijective correspondence between matrices $A \in M_{m\times n}(\mathbb{R})$ and bounded linear transformations $T \in B(\mathbb{R}^n, \mathbb{R}^m)$.
- Let X be a real normed space. The space $X' := B(X, \mathbb{R})$ is called the **dual of** X; its elements are **bounded linear functionals** on X. If X is complex, $B(X, \mathbb{C})$ is its dual.[1]
- The dual of $\mathbb{R}$ is $\mathbb{R}$: each bounded linear functional $T \in B(\mathbb{R}, \mathbb{R})$ is realized by multiplication with a real constant:

$$T \in B(\mathbb{R}, \mathbb{R}) \quad \Longleftrightarrow \quad Tx = \lambda x, \quad \lambda \in \mathbb{R}.$$

- **Riesz representation theorem**: Let $L_2(I, \mathbb{R})$ be the space of real-valued square-integrable functions on an open interval $I \subset \mathbb{R}$, with norm

$$\|f\|_{L_2(I,\mathbb{R})} = \Big(\int_I |f(t)|^2 \, dt \Big)^{1/2}.$$

The Riesz representation theorem asserts that each bounded linear functional T on $L_2(I, \mathbb{R})$ can be identified with an element $g \in L_2(I, \mathbb{R})$, via

$$Tf = \int_I f(t)g(t) \, dt.$$

Boundedness and continuity

Continuity

Recall that a mapping $f\colon X \to Y$ between two metric spaces is said to be **continuous at** x_0 if

$$f(x_n) \to f(x_0) \ \text{ in } Y \qquad \text{as} \qquad x_n \to x_0 \ \text{ in } X.$$

Since continuous and sequential limits agree, this is the same as

$$\forall\, \varepsilon > 0 \quad \exists\, \delta > 0; \qquad d_Y(f(x), f(x_0)) < \varepsilon \quad \text{for} \quad d_X(x, x_0) < \delta.$$

A mapping that is continuous at all points in X is called **continuous**.

Ex.
In a normed space, $(X, \|\cdot\|)$, the norm is a continuous function $X \to \mathbb{R}$: if $x_n \to x_0$ in X, then

$$d_{\mathbb{R}}(\|x_n\|, \|x_0\|) = \big|\|x_n\| - \|x_0\|\big| \le \|x_n - x_0\| = d_X(x_n, x_0) \to 0,$$

by the reverse triangle inequality.

℘ **For linear operators, continuity means boundedness**
Let $T \in L(X, Y)$. Then the following statements are equivalent:

- T is everywhere continuous.
- T is continuous at $x = 0$.
- T is bounded.

[1] This notion of dual coincides with that of a **continuous dual**; it is possible to define more general duals.

Proof
First, note that for any fixed $x_0 \in X$,

$$Tx_n \to Tx_0 \quad \text{as} \quad x_n \to x_0 \quad \Longleftrightarrow \quad T(x_n - x_0) \to \mathbf{0}_Y \quad \text{as} \quad (x_n - x_0) \to \mathbf{0}_X$$
$$\overset{z_n = x_n - x_0}{\Longleftrightarrow} \quad Tz_n \to \mathbf{0}_Y \quad \text{as} \quad z_n \to \mathbf{0}_X,$$

so that, for linear operators, continuity at the origin is the same as continuity everywhere (x_0 is arbitrary).

To see that boundedness and continuity at the origin are equivalent, assume first that T is bounded. Then

$$\|Tx\|_Y \leq \|T\|\|x\|_X \to 0 \quad \text{as} \quad \|x\|_X \to 0,$$

so that T is also continuous. Contrariwise, assume that T is continuous at the origin. Then

$$\|Tx\|_Y = \|Tx - T\mathbf{0}\|_Y \leq \varepsilon \quad \text{for} \quad \|x\|_X = \|x - \mathbf{0}\|_X \leq \delta.$$

But T is linear, so by scaling x (replace x with δx) we obtain

$$\|Tx\|_Y \leq \frac{\varepsilon}{\delta} \quad \text{for} \quad \|x\|_X \leq 1.$$

Thus T is bounded.

Ex. Any linear operator $T \in L(X, Y)$ defined on a *finite-dimensional* normed space X is continuous. Reason: identify $X \cong \mathbb{R}^n$ and note that

$$\operatorname{ran}(T) = \operatorname{span}\{Te_1, \ldots, Te_n\} \cong \mathbb{R}^m \quad \text{for some } m \leq n,$$

where $\{e_1, \ldots, e_n\}$ is a basis for $\mathbb{R}^n$. Hence, $T\colon X \cong \mathbb{R}^n \to \mathbb{R}^m \cong \tilde{Y} \subset Y$ is a linear transformation onto a finite-dimensional subspace $\tilde{Y}$ of Y, and, as such, has a matrix representation

$$T\colon x \mapsto Ax = \big(\sum_{j=1}^{n} a_{ij}x_j\big)_{i=1}^{m}.$$

All norms on a finite-dimensional vector space are equivalent, so whatever the norms of X and Y, we can consider any suitable norms for $\mathbb{R}^n \cong X$ and $\mathbb{R}^m \cong \tilde{Y}$. Choose, for example, the l_∞-norm: then

$$\|Ax\|_{l_\infty} = \max_{1 \leq i \leq m} \Big|\sum_{j=1}^{n} a_{ij}x_j\Big| \leq n \max_{i,j} |a_{ij}| \max_j |x_j| = n \max_{i,j} |a_{ij}| \|x\|_{l_\infty}.$$

This means that T is bounded with $\|T\| \leq n \max_{i,j} |a_{ij}|$, and therefore also continuous.
N.b. Equivalent norms yield the same open and closed sets, the same convergence, but *not the same constants* in the estimates – in particular, the exact value of $\|T\|$ depends on the norms for X and Y.

℘ **The kernel of a bounded operator is closed**
Let $T \in B(X, Y)$. Then $\ker(T)$ is a closed subspace of X. In particular, if X is a Banach space, so is $\ker(T)$.

Proof
Take

$$\{x_n\}_{n\in N} \subset \ker(T); \qquad \lim_{n\to\infty} x_n = x_0 \in X.$$

We want to show that $x_0 \in \ker(T)$. But this follows from the continuity of T:

$$\|Tx_0\|_Y = \|Tx_0 - Tx_n\|_Y \leq \|T\|\|x_0 - x_n\|_X \to 0 \quad \text{as} \quad x_n \to x_0 \qquad \Longrightarrow \qquad Tx_0 = 0.$$

If, furthermore, X is complete, so is the closed subspace $\ker(T) \subset X$.

Ex.

- The null space of a matrix $A \in M_{m\times n}(\mathbb{R})$ is a closed subspace of $\mathbb{R}^n$.
- In $L_2((-\pi,\pi),\mathbb{R})$, the kernel of the bounded linear functional

$$T\colon f \mapsto \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(t)\,dt$$

is a closed subspace; it consists of all functions with zero Fourier coefficient before $\sin(t)$ in its Fourier expansion.

N.b. It is not always true that the range of a bounded linear operator is closed.

4.6 Bounded linear operators on Hilbert spaces

Let H be a Hilbert space, as before our main examples are $H = \mathbb{R}^n$ and $H = \mathbb{C}^n$.

Bounded functionals on Hilbert spaces

The Riesz representation theorem

A Hilbert space is its own dual: every bounded linear functional $T \in B(H,\mathbb{K})$ is given by an inner product,

$$Tx = \langle x, z\rangle,$$

for a unique $z \in H$. Moreover, $\|T\|_{B(H,\mathbb{K})} = \|z\|_H$.

N.b. Note that any function $x \mapsto \langle x, y\rangle$ defines a bounded linear functional on H.

Proof
Existence: Let

$$N = \ker(T).$$

Then N is a closed linear subspace of H. If $N = H$, we have $T = 0$ in $B(H,\mathbb{F})$ and $Tx = \langle x, 0\rangle$.

Assume now that $N \neq H$. According to the corollary of the projection theorem, there exists $z_0 \in N^\perp$, $z_0 \neq 0$. Since $z_0 \perp \ker(T)$ we have $Tz_0 \neq 0$. Consequently,

$$x - \frac{Tx}{Tz_0}z_0 \in \ker(T) \quad \text{for all} \quad x \in H,$$

implying

$$\Big\langle x - \frac{Tx}{Tz_0}z_0, z_0\Big\rangle = 0 \;\Leftrightarrow\; Tx\Big\langle \frac{1}{Tz_0}z_0, z_0\Big\rangle = \langle x, z_0\rangle \;\Leftrightarrow\; Tx = \frac{Tz_0}{\|z_0\|^2}\langle x, z_0\rangle = \Big\langle x, \frac{\overline{Tz_0}}{\|z_0\|^2}z_0\Big\rangle.$$

Thus

$$Tx = \langle x, z\rangle \quad \text{for} \quad z := \frac{\overline{Tz_0}}{\|z_0\|^2} z_0.$$

Uniqueness: If, in addition,

$$Tx = \langle x, w\rangle \quad \text{for all} \quad x \in H,$$

then

$$\langle x, z - w\rangle = Tx - Tx = 0 \quad \text{for all} \quad x \in H,$$

so that $z = w$.

Equality of norms: We have

$$\|T\| = \sup_{\|x\|=1} |Tx| = \sup_{\|x\|=1} |\langle x, z\rangle| \le \sup_{\|x\|=1} \|x\|\|z\| = \|z\|,$$

by the Cauchy–Schwarz inequality. Contrariwise,

$$\|z\|^2 = \langle z, z\rangle = |Tz| \le \|T\|\|z\| \quad \Longrightarrow \quad \|z\| \le \|T\|.$$

Thus $\|T\| = \|z\|$.

Ex.

- $\mathbb{C}^n$ is its own dual: every bounded linear functional on $\mathbb{C}^n$ is realized by a dot product: $Tx = x \cdot \overline{y}$.
- $L_2(\mathbb{R}, \mathbb{C})$ is its own dual: every bounded linear functional on $L_2(\mathbb{R}, \mathbb{C})$ is realized by an inner product: $Tf = \int_{\mathbb{R}} f(s)\overline{g(s)}\, ds$. By the Cauchy–Schwarz inequality,

$$|Tf| = |\langle f, g\rangle| = \Big| \int_{\mathbb{R}} f(s)\overline{g(s)}\, ds \Big| \le \Big(\int_{\mathbb{R}} |f(s)|^2\, ds \Big)^{1/2} \Big(\int_{\mathbb{R}} |g(s)|^2\, ds \Big)^{1/2} = \|f\|\|g\|,$$

with equality for $f = \lambda g$; hence, $\|T\| = ||g\|$.

Suppose that X is an inner-product space, $X \subset H$, where H is a Hilbert space and X is dense in H, i.e. $\operatorname{clos}(X) = H$, this is the case when H is the completion of X. Then any bounded linear functional $T : X \to \mathbb{R}(\mathbb{C})$ can be extended uniquely to a functional on H, thus by the Riesz representation theorem

$$T(x) = \langle x, h\rangle, \quad h \in H,$$

note that now $h \in H$ but not necessarily in X.

Ex. Let $X = C([0, 1], \mathbb{C})$ with the inner product $\langle f, g\rangle = \int_0^1 f(t)\overline{g(t)}dt$. Then the completion of X is $L^2([0, 1], \mathbb{C})$. Consider

$$T : f \mapsto \int_{1/2}^1 f(t)dt.$$

Then T is a bounded functional,

$$Tf = \langle f, h\rangle, \quad h(t) = \begin{cases} 0, & 0 \le t < 1/2 \\ 1, & 1/2 \le t \le 1. \end{cases}$$

Here $h \in L^2([0, 1], \mathbb{C})$ but $h \notin C([0, 1], \mathbb{C})$.

Adjoints

Let $T \in B(H)$ be a bounded linear operator $H \to H$.

- The **adjoint** of T is the operator $T^* \in B(H)$ defined by

$$\langle Tx, y\rangle = \langle x, T^* y\rangle \quad \text{for all} \quad x, y \in H.$$

- T is called **self-adjoint** if $T = T^*$.

Properties of the adjoint

The adjoint is well defined: for each $T \in B(H)$, there exists a unique $T^* \in B(H)$. The map $^*\colon B(H) \to B(H)$, $T \mapsto T^*$ satisfies the following properties:

- It is anti-linear: $(\mu S + \lambda T)^* = \overline{\mu} S^* + \overline{\lambda} T^*$, for all $S, T \in B(H)$ and $\mu, \lambda \in \mathbb{K}$.
- It is bounded with unit norm: $\|T^*\| = \|T\|$.
- It is invertible with itself as inverse: $(T^*)^* = T$.

N.b. We adopt the convention that $T^{**} \overset{\text{def.}}{=} (T^*)^*$.

Proof

Existence of the adjoint: For each $y \in H$, the map $x \mapsto \langle Tx, y\rangle$ is a bounded linear functional, since by the Cauchy–Schwarz inequality and the boundedness of T,

$$|\langle Tx, y\rangle| \le \|Tx\|\|y\| \le \|T\|\|x\|\|y\|.$$

The Riesz representation theorem thus guarantees that there exists unique $y^* \in H$ with

$$\langle Tx, y\rangle = \langle x, y^*\rangle.$$

Define $T^* y := y^*$.

Linearity and boundedness of the adjoint: Since $\lambda y_1^* + \mu y_2^*$ is the unique element corresponding to the functional $\langle Tx, \lambda y_1 + \mu y_2\rangle$, the map $y \mapsto T^* y$ is linear. It is also bounded: we have

$$|\langle x, T^* y\rangle| = |\langle Tx, y\rangle| \le \|T\|\|x\|\|y\|,$$

so by choosing $x = T^* y$ we obtain

$$\|T^* y\|^2 \le \|T\|\|T^* y\|\|y\| \quad \Longrightarrow \quad \|T^* y\| \le \|T\|\|y\|,$$

so that $\|T^*\| \le \|T\|$. But $T^{**} = T$, so we also obtain that $\|T\| \le \|T^*\|$, whence $\|T^*\| = \|T\|$.

Anti-linearity and boundedness of *: Since

$$\begin{aligned}\langle x, (\mu S + \lambda T)^* y\rangle = \langle (\mu S + \lambda T)x, y\rangle = \mu\langle Sx, y\rangle + \lambda\langle Tx, y\rangle = \\ \mu\langle x, S^* y\rangle + \lambda\langle x, T^* y\rangle = \langle x, (\overline{\mu} S^* + \overline{\lambda} T^*) y\rangle,\end{aligned}$$

the map * is anti-linear:

$$(\mu S + \lambda T)^* = \overline{\mu} S^* + \overline{\lambda} T^*.$$

In view of that $\|T\| = \|T^*\|$ it follows that * is bounded with norm 1.

Invertibility of *:

$$\langle Tx, y\rangle = \langle x, T^* y\rangle = \overline{\langle T^* y, x\rangle} = \overline{\langle y, T^{**} x\rangle} = \langle T^{**} x, y\rangle,$$

so that $\langle (T - T^{**})x, y\rangle = 0$ for all $x, y \in H$. Choose $y = (T - T^{**})x$. Then

$$\|(T - T^{**})x\|^2 = 0 \quad \text{for all} \quad x \in H,$$

meaning that $T = T^{**}$ in $B(H)$.

Ex.

- If $T : \mathbb{C}^n \to \mathbb{C}^n$ is defined by a $n \times n$ matrix A then

$$\langle Tx, y\rangle = Ax \cdot y = (Ax)^t \overline{y} = x^t A^t \overline{y} = x \cdot A^* y,$$

  where $A^* = \overline{A^t}$ is the conjugate transpose of A.
- Let $T : l^2(\mathbb{C}) \to l^2(\mathbb{C})$ be defined by $(Tx)_j = c_j x_j$, where c_j is a bounded sequence of complex numbers. Then

$$\langle Tx, y\rangle = \sum_1^\infty c_j x_j \overline{y_j} = \langle x, T^* y\rangle, \quad (T^* y)_j = \overline{c_j} y_j.$$

  Thus T is self-adjoint if and only if $c_j \in \mathbb{R}$ for all j.

For matrices, we extend the definition of adjoint to rectangular matrices.

- For $A \in M_{m\times n}(\mathbb{C})$ the matrix $A^* \in M_{n\times m}(\mathbb{C})$ defined by $a^*_{ij} := \overline{a_{ji}}$ is called its **adjoint** or **conjugate transpose**. Equivalently,

$$A^* = \overline{A^t},$$

  where A^t is the transpose of A.
- A square matrix A is said to be **Hermitian** if $A = A^*$.

Self-adjoint matrices are symmetric or Hermitian

- If $T \in B(\mathbb{R}^n)$ is realized by a matrix $A \in M_{n\times n}(\mathbb{R})$, T is self-adjoint if and only if A is symmetric.
- If $T \in B(\mathbb{C}^n)$ is realized by a matrix $A \in M_{n\times n}(\mathbb{C})$, T is self-adjoint if and only if A is Hermitian.

Ex. Let T_θ be a rotation in $\mathbb{R}^2$ by θ radians counter clock-wise, it is given by the matrix

$$A_\theta = \begin{bmatrix} \cos\theta & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

Then the adjoint operator is given by $(A_\theta)^* = A_{-\theta}$, it is the rotation $T_{-\theta}$. The rotation is self-adjoint if and only if $A_\theta = A_{-\theta} \iff \theta = k\pi, k \in \mathbb{Z}$.

The adjoint of the composition and normal operators

Let $S, T \in B(H)$ then $(S \circ T)^* = T^* \circ S^*$. It follows directly from the definition

$$\langle STx, y\rangle = \langle Tx, S^* y\rangle = \langle x, T^* S^* y\rangle.$$

For matrices this implies $(AB)^t = B^t A^t$.

Then for any $T \in B(H)$ the operators T^*T and TT^* are self-adjoint since $T^{**} = T$. An operator $T \in B(H)$ is called **normal** if $T^*T = TT^*$. In particular, any self-adjoint operator is normal.

Unitary operators and orthogonal matrices

- An operator $U \in B(H)$ is called **unitary** if $UU^* = U^*U = \mathrm{Id}$. Similarly, a matrix $A \in M_{n\times n}(\mathbb{K})$ is called **unitary** if the corresponding operator is unitary, i.e., if $A^*A = I$.
- A unitary real matrix $Q \in M_{n\times n}(\mathbb{R})$ is called **orthogonal**.
- Each unitary operator is normal.

Ex.

- Let T_θ be a rotation in $\mathbb{R}^2$ by θ radians counter clock-wise, the adjoint is $T_{-\theta}$ and $T_\theta T_{-\theta} = T_{-\theta}T_\theta = I$. Thus the rotation matrix is unitary.
- Let $T : l^2(\mathbb{C}) \to l^2(C)$ be given by $(Tx)_j = c_j x_j,\ c_j \in \mathbb{C}$. Then $(T^*x)_j = \overline{c_j}x_j$ and

$$(TT^*x)_j = (T^*Tx)_j = |c_j|^2 x_j.$$

Thus T is always a normal operator. It is unitary if and only if $|c_j| = 1$ for each j.

Unitary operators preserve inner products

If $U \in B(H)$ is unitary, then

$$\langle Ux, Uy\rangle = \langle x, y\rangle, \quad \text{for all} \quad x, y \in H.$$

In particular, U is an isometry.

Proof

$$\langle Ux, Uy\rangle = \langle x, U^*Uy\rangle = \langle x, y\rangle.$$

Orthogonal matrices describe orthonormal bases

The columns $\{Q_1, \dots, Q_n\}$ of an orthogonal (unitary) matrix Q is an orthonormal basis for $\mathbb{R}^n$ ($\mathbb{C}^N$).

N.b. The same is true for the rows of Q.

Proof

Since the rows of Q^* are columns of $\overline{Q}$, we have

$$Q^*Q = ((Q_i)_i)^*(Q_j)_j = (\overline{Q_i} \cdot Q_j)_{ij} = I$$

if and only if $Q_i \perp Q_j$ for $i \neq j$, and $|Q_i| = 1$.

4.7 Functional calculus

Limits of bounded linear operators

℘ B(X,Y) is Banach for Y Banach

If Y is complete, so is $B(X, Y)$.

N.b. Note that X has no role in the completeness of $B(X, Y)$.

Proof

Pointwise convergence. Let $\{T_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $B(X, Y)$. Then, for each fixed $x \in X$,

$$\|(T_n - T_m)x\|_Y \leq \|T_m - T_n\|\|x\|_X \overset{m,n\to\infty}{\to} 0,$$

so that $\{T_n x\}_{n\in\mathbb{N}}$ is Cauchy in Y. By assumption, Y is complete, so $\{T_n x\}_{n\in\mathbb{N}}$ is convergent. Define

$$Tx := \lim_{n\to\infty} T_n x, \qquad x \in X.$$

The pointwise limit defines a linear and bounded transformation. With this construction $T\colon X \to Y$ is linear,

$$T(\lambda x + \mu y) = \lim_{n\to\infty} T_n(\lambda x + \mu y) = \lim_{n\to\infty} (\lambda T_n x + \mu T_n y) = \lambda \lim_{n\to\infty} T_n x + \mu \lim_{n\to\infty} T_n y = \lambda Tx + \mu Ty,$$

and for each fixed $x \in X$ there exists n_ε (depending also on x), such that, for all $n \geq n_\varepsilon$,

$$\|Tx\|_Y \leq \|(T - T_n)x\|_Y + \|T_n x\|_Y \leq \varepsilon + \|T_n x\|_Y \leq \varepsilon + \underbrace{\sup_{n\in\mathbb{N}} \|T_n\|}_{\text{finite}} \|x\|_X,$$

where we have used that Cauchy sequences are bounded (so that $\|T_n\|$ is bounded, uniformly for $n \in \mathbb{N}$). By taking the supremum over all x with $\|x\|_X = 1$, we obtain that T is bounded.

Convergence in $B(X, Y)$. It remains to show that $T_n \to T$ in $B(X, Y)$. Similarly to the above argument, if $m \geq n_{\varepsilon/2}$ (depending also on x), we have

$$\begin{aligned}\|(T - T_n)x\|_Y \leq \|(T - T_m)x\|_Y + \|(T_m - T_n)x\|_Y &\leq \tfrac{\varepsilon}{2} + \|(T_m - T_n)x\|_Y \\ &< \tfrac{\varepsilon}{2} + \|T_m - T_n\|\|x\|_X.\end{aligned}$$

Since $\{T_n\}_{n\in\mathbb{N}}$ is Cauchy, there exists $N_{\varepsilon/2}$ such that

$$\|T_n - T_m\| < \frac{\varepsilon}{2} \quad \text{for} \quad m, n \geq N_{\varepsilon/2}.$$

Choose $n \geq N_{\varepsilon/2}$ and, for each x, an appropriate $m \geq \max\{n_{\varepsilon/2}, N_{\varepsilon/2}\}$ By taking the supremum over all x with $\|x\|_X = 1$ we thus find

$$\|T - T_n\| < \varepsilon \quad \text{for} \quad n \geq N_{\varepsilon/2}.$$

Hence, $\lim_{n\to\infty} T_n = T$ in $B(X, Y)$.

Powers and power series

Let $T \in B(X)$ be a bounded linear operator on a normed space X. Then $T^2 = T \circ T$ is also a bounded linear operator and $\|T^2\| \leq \|T\|^2$ since

$$\|T^2 x\| \leq \|T\|\|Tx\| \leq \|T\|\|T\|\|x\| = \|T\|^2\|x\|.$$

Similarly, one may define $T^k = T^{k-1} \circ T$ and show that $\|T^k\| \leq \|T\|^k$.

If we assume that X is a Banach space then $B(X)$ is also a Banach space and we may consider sequences and series of operators. A bounded operator $T \in B(X)$ is called **invertible** if there exists $S \in B(X)$ such that $T \circ T$ and $T \circ S$ are the identity operators. Similarly, $T \in B(X, Y)$ is invertible if there exists $S \in B(Y, X)$ such that $S \circ T = I_X$ and $T \circ S = I_Y$, S is called the inverse operator, it is unique.

℘ **Operator** $(I - T)^{-1}$
Suppose that $T \in B(X)$ and $\|T\| < 1$ then $(I - T)$ is invertible and the series

$$\sum_{j=0}^{\infty} T^j$$

defines the inverse operator which is bounded.

Proof
Since $\|Tx\| \leq q\|x\| < \|x\|$, it is clear that $\ker(I-T) = \{\mathbf{0}\}$. Thus $I-T$ is injective.

Let $q = \|T\| < 1$ then $\|T^k\| \leq q^k$, $k = 1, 2, ...$ Let $S_n = \sum_{j=0}^{n} T^j$, then for $n, m > k$

$$\|S_n - S_m\| \leq \sum_{k}^{\infty} \|T^j\| \leq \sum_{k}^{\infty} q^j = \frac{q^k}{1-q} \to 0 \quad \text{as } k \to \infty.$$

Thus $\{S_n\}$ is a Cauchy sequence in $B(X)$ and since $B(X)$ is complete it converges to some $S \in B(X)$. Further, $S = S_n + R_n$, where $\|R_n\| \leq q^n(1-q)^{-1}$. Then

$$\begin{aligned}(I-T)S = (I-T)(S_n + R_n) &= \sum_{k=0}^{n}(I-T)T^k + (I-T)R_n \\ &= \sum_{k=0}^{n}(T^k - T^{k+1}) + (I-T)R_n = I - T^{n+1} + (I-T)R_n.\end{aligned}$$

It implies $(I-T)S - I = -T^{n+1} + (I-T)R_n$ and

$$\|(I-T)S - T\| \leq q^{n+1} + (1+q)q^n(1-q)^{-1} \to 0, \quad n \to \infty.$$

Hence $(I-T)S = I$. Thus $(I-T)$ is surjective and S is the inverse of $I-T$, also $S(I-T) = I$.

Clearly, $\|S\| \leq \sum_k q^k = (1-q)^{-1}$.

The exponential map
Suppose that $T \in B(X)$, where X is a Banach space, then the map

$$\exp(T) \overset{\text{def.}}{=} \sum_{j=0}^{\infty} \frac{T^j}{j!} = I + T + \frac{T^2}{2} + \frac{T^3}{3!} + \dots,$$

also written e^T, is called the **exponential** of T.

℘ **The exponential map is well defined**
Let $T \in B(X)$ then $\exp(T) \in B(X)$ and $\|\exp(T)\| \leq e^{\|T\|}$.

Proof
The statement $\exp(T) \in L(B)$ is equivalent to

$$\lim_{N\to\infty} \underbrace{\sum_{j=0}^{N} \frac{T^j}{j!}}_{=:y_N}$$

is well-defined as a limit in $B(X)$. For $N \geq m$ we have

$$\|y_N - y_m\| \leq \sum_{j=m+1}^{N} \frac{\|T^j\|}{j!} \leq \sum_{j=m+1}^{\infty} \frac{\|T\|^j}{j!} \to 0,$$

as $N \geq m \to \infty$. Hence $\{y_N\}_N$ is Cauchy and converges in $B(\mathbb{C}^n)$. The same argument without y_m shows that

$$\|\exp(T)\| \leq e^{\|T\|}.$$

Properties of the exponential map
Let $S, T \in B(X)$

- If $ST = TS$, then

$$S\exp(T) = \exp(T)\,S \qquad \text{and} \qquad \exp(S+T) = \exp(S)\exp(T).$$

- If S is invertible, then

$$S\exp(T)S^{-1} = \exp(STS^{-1}).$$

- $\exp(T)$ is invertible with

$$\big(\exp(T)\big)^{-1} = \exp(-T).$$

Nilpotent operators

A linear operator $T \in L(X)$ is called **nilpotent** if there exists a positive integer p such that $T^p = \mathbf{0}$.

Ex.

- For example the differentiation operator $T = \frac{d}{dt}$ on the space $P_n(\mathbb{R})$ of polynomials of degree not greater than n is nilpotent since $T^{n+1} = 0$.
- The following square matrix defines a nilpotent operator on $\mathbb{R}^n$

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

The power series of the operators simplifies when T is nilpotent and becomes finite sums. For example if $T^p = 0$ then

$$\exp(T) = \sum_{j=0}^{p-1} \frac{T^j}{j!}.$$

Existence theory for constant-coefficient linear ODE's

Consider

$$\dot{u} = Au, \qquad u(0) = u_0 \in \mathbb{C}^n. \qquad (1)$$

(The choice $t_0 = 0$ is irrelevant, since $u(\cdot - t_0)$ is a solution exactly if u is.) Note that the right-hand side $f(u) = Au$ is uniformly Lipschitz with

$$|Au - Av| \le \|A\||u - v|, \qquad \|A\| = \sup_{|u|=1} |Au|,$$

so that this problem is locally and uniquely solvable. As we shall see, any solution can be globally continued on $\mathbb{R}$, and even explicitly constructed.

$\wp$ Characterization of solution spaces

The solution set of $\dot{u} = Au$ is a vector space isomorphic to $\mathbb{C}^n$. If A is real and only real initial data $u_0 \in \mathbb{R}^n$ is considered, then the solution space is isomorphic to $\mathbb{R}^n$.

Proof

By the Picard–Lindelöf theorem, the solution map $u_0 \overset{\varphi}{\mapsto} u(\cdot; u_0)$ is well defined.

Injectivity. If $\varphi(u_0) = \varphi(v_0)$ are two identical solutions, then clearly $\varphi(u_0)(0) = \varphi(v_0)(0)$, meaning $u_0 = v_0$.

Surjectivity. The map φ is surjective onto the set of solutions: any solution v of $\dot{v} = Av$ gives rise to initial data $v_0 := v(0)$, which in turn generates a solution $u(\cdot; v_0)$. By uniqueness $v = u = \varphi(v_0)$, so that $v \in \operatorname{ran}(\varphi)$.

Linearity. φ is linear: if v solves (1) with $v(0) = v_0$, and w solves (1) with $w(0) = w_0$, then $u = \lambda v + \mu w$ solves (1) with $u(0) = \lambda v_0 + \mu w_0$.

Conclusion: Thus $\varphi\colon \mathbb{C}^n \to \varphi(\mathbb{C}^n)$ is a vector space isomorphism onto its image, which consequently is a complex vector space of dimension n. Since we have shown that the image $\varphi(\mathbb{C}^n)$ consists of all solutions of $\dot{u} = Au$, the proposition follows.

Fundamental matrix

A basis $\{u_j\}_{j=1}^n$ of solutions is called a **fundamental system** for $\dot{u} = Au$; the corresponding matrix $(u_j)_j$ is a **fundamental matrix**.

N.b. According to the above characterization, a set of solutions $\{u_j\}_j$ is a fundamental system exactly if $\{u_j(0)\}_j$ is a basis for $\mathbb{C}^n$ (or $\mathbb{R}^n$ if we are considering real solutions).

Solution formula

The unique solution of (1) is

$$u(t; u_0) = \exp(tA)u_0,$$

and $\exp(tA)$ is a fundamental matrix with $\exp(tA)|_{t=0} = I$.

Proof

Since

$$\frac{d}{dt}\exp(tA) = A\exp(tA),$$

$\exp(tA)$ solves the matrix equation $\dot{X} = AX$. This means that each column of $\exp(tA)$ solves $\dot{u} = Au$. Since $\exp(tA)$ is invertible, the columns are linearly independent, so they must span the solution space (it is n-dimensional, as we have proved). Thus $\exp(tA)$ is a fundamental matrix. That $\exp(\mathbf{0}) = \exp(tA)|_{t=0} = I$ follows immediately from the definition of the exponential map.

Now, multiplying $\exp(tA)$ from the right with u_0 yields the solution of the initial-value problem (1), since

$$\frac{d}{dt}\exp(tA)u_0 = A\exp(tA)u_0,$$

and $u(0) = \exp(tA)|_{t=0}u_0 = Iu_0 = u_0$.

4.8 Spectral theory in finite dimensional spaces

Let $A \in M_{n\times n}(\mathbb{C})$ be the realization of a bounded linear transformation $\mathbb{C}^n \to \mathbb{C}^n$ You can think of A as real, but, if so, still describing a bounded linear map $\mathbb{C}^n \to \mathbb{C}^n$. The aim of this section is to compute the $\exp(tA)$.

- Suppose that A is similar to a diagonal matrix, $A = QDQ^{-1}$, where $D = \operatorname{diag}(\lambda_1, ..., \lambda_n)$ then $A^k = QD^kQ^{-1}$ and $\exp(A) = Q\exp(D)Q^{-1}$, where $\exp(D) = \operatorname{diag}(e^{\lambda_1}, ..., e^{\lambda_n})$.

- If A is nilpotent, i.e., if

$$A^{n_0} = 0 \quad \text{for some } n_0 \in \mathbb{N},$$

then $\exp(A)$—and therefore $\exp(tA)$—is a finite sum:

$$\exp(A) = \sum_{j=0}^{\infty} \frac{A^j}{j!} = \sum_{j=0}^{n_0-1} \frac{A^j}{j!} = I + A + \ldots + \frac{A^{n_0-1}}{(n_0-1)!}.$$

In general, other methods must be employed.

The spectrum of an operator

Let $T \in B(X)$ be a bounded linear transformation $X \to X$ (for example, $T\colon \mathbb{C}^n \to \mathbb{C}^n$ given by A).

- $\lambda \in \mathbb{C}$ is called an **eigenvalue** of T if there exists a nonzero $v \in X$ such that

$$Tv = \lambda v.$$

The vector v is called an **eigenvector** corresponding to the eigenvalue λ.

- The set of values $\lambda \in \mathbb{C}$ for which $(T - \lambda I)$ is invertible with a bounded inverse $(T - \lambda I)^{-1} \in B(X)$ is called the **resolvent set** of T. Its complement in $\mathbb{C}$, denoted $\sigma(T)$, is called the **spectrum** of T.
- Clearly each eigenvalue $\lambda \in \sigma(T)$, the set of all eigenvalues is called the **point spectrum** of T. In general $\sigma(T)$ is larger than the set of eigenvalues, it contains also λ such that $T - \lambda I$ is one-to-one but has no inverse in $B(X)$.

Ex.
- The formula for the inverse of $I - T$ can be rewritten for $(T - \lambda I)^{-1}$ when $|\lambda| > \|T\|$. It implies that for a bounded operator T its spectrum $\sigma(T)$ is contained in the closed disk $\{z \in \mathbb{C}; |z| \leq \|T\|\}$.
- Let $T : l^2 \to l^2$ be the left shift. Then T is injective but it has no inverse. Thus $0 \in \sigma(T)$ but it is not an eigenvalue.

$\wp$ Self-adjoint operators have real spectrum

Let H be a Hilbert space and $T \in B(H)$ be a self-adjoint operator. Then the eigenvalues of T are real, and eigenspaces corresponding to different eigenvalues are orthogonal.

Proof

If $T = T^*$, then

$$\langle Tx, x\rangle = \langle x, Tx\rangle = \overline{\langle Tx, x\rangle} \in \mathbb{R} \quad \text{for all } x \in H.$$

Hence, if $\mu, \lambda \in \mathbb{C}$, $\mu \neq \lambda$, are eigenvalues of T, with eigenvectors x, y, respectively, then

$$\mu\|x\|^2 = \langle \mu x, x\rangle = \langle Tx, x\rangle \in \mathbb{R},$$

so that $\mu \in \mathbb{R}$ (and, similarly, $\lambda \in \mathbb{R}$). Then

$$(\mu - \lambda)\langle x, y\rangle = \langle \mu x, y\rangle - \langle x, \lambda y\rangle = \langle Tx, y\rangle - \langle x, Ty\rangle = 0,$$

since T is self-adjoint. Thus $x \perp y$.

Spectrum of an operator in finite dimensional space

For matrices, the spectrum consists only of eigenvalues

For $A \in M_{n\times n}(\mathbb{C})$,

$$\sigma(A) = \{\lambda \in \mathbb{C} \colon \det(A - \lambda I) = 0\}$$

consists of the roots of the **characteristic polynomial** $p_A(\lambda) \overset{\text{def.}}{=} \det(A - \lambda I)$; these are the eigenvalues of A.

N.b. Defining properties of the determinant are not treated in this course. We use the fact from linear algebra that a square matrix is invertible if and only if its determinant is non-zero.

Proof

Since $\mathbb{C}^n$ is finite-dimensional, we have that

$$\begin{aligned} \exists v \neq 0; \quad (A-\lambda)v = 0 \quad &\iff \quad \ker(A - \lambda I) \text{ nontrivial} \\ &\iff \quad (A - \lambda I) \text{ not invertible} \quad \iff \quad \det(A - \lambda I) = 0. \end{aligned}$$

The proposition then follows by noting that $\det(A - \lambda I)$ is a polynomial in λ of degree n (which, according to the fundamental theorem of algebra, has n roots counting multiplicity).

Ex.

- If A and B are two similar matrices then $p_A(\lambda) = p_B(\lambda)$. Thus a linear operator in a finite dimensional vector space has a characteristic polynomial that does not depend on the choice of a basis.
- If A is a hermitian (symmetric real-valued) matrix, then it defines a self-adjoint operator. All its eigenvalues are real and eigenvectors corresponding to distinct eigenvectors are orthogonal.

Multiplicity

- The multiplicity of a root λ of $p_A(\lambda)$ is the **algebraic multiplicity** of the the eigenvalue λ, denoted $\text{mult}(\lambda)$.
- The eigenvectors corresponding to an eigenvalue λ span a subspace of $\mathbb{C}^n$,

  $$\ker(A - \lambda I),$$

  called the **eigenspace** of λ. The dimension of this space is the **geometric multiplicity** of λ.
- An eigenvalue λ is called **simple** if λ is simple as a root of $p_A(\lambda)$,

  $$\lambda \text{ simple} \quad \overset{\text{def}}{\iff} \quad \text{mult}(\lambda) = 1;$$

  it is **semi-simple** if the geometric and algebraic multiplicity coincide,

  $$\lambda \text{ semi-simple} \quad \overset{\text{def}}{\iff} \quad \text{mult}(\lambda) = \dim \ker(A - \lambda I).$$

Ex.

- Let $A = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}$. Then $p_A(\lambda) = (a-\lambda)^2$ and a is an eigenvalue mult$(a) = 2$. The solutions to $Ax = ax$ are $x = (x_1, 0)$, thus the corresponding eigenspace has dimension one and the geometric multiplicity is 1.
- If D is a diagonal matrix and A is similar to D,

$$D = \text{diag}(\lambda_1, ...\lambda_n), \quad A = QDQ^{-1},$$

then the algebraic multiplicity of each eigenvalue of D is equal to its geometric multiplicity, it is the number of times this eigenvalue appears on the diagonal. Further,

$$p_A = p_D \quad \text{and} \quad Q(E_{\lambda,A}) = E_{\lambda,D}.$$

Thus A and D have the same eigenvalues with equal algebraic and geometric multiplicities; all eigenvalues of A are semi-simple.

Invariant subspaces and the Caley-Hamilton theorem

Let T be a linear operator on a vector space V.

Invariant subspaces

A subspace W of V is called a T-**invariant subspace** if $T(W) \subset W$.

Ex.

- For any linear operator T the following subspaces are invariant: $\ker(T)$, $\text{ran}(T)$.
- If λ is an eigenvalue of T then E_λ is T-invariant subspace.
- Let $T \in L(V)$ and $x \in V$, define

$$W = \text{span}(x, T(x), T^2(x), ...).$$

Then W is an invariant subspace of V, it is called the T-**cyclic subspace generated by** x.

℘ Cyclic subspaces and characteristic polynomials

Let V be a finite dimensional vector space and $T \in L(V)$. Suppose that W is the T-cyclic subspace generated by x, $k = \dim(W)$ and let $S \in L(W)$ be the restriction of T onto W. Then
(i)

$$\{x, T(x), ..., T^{k-1}(x)\}$$

is a basis for W.
(ii) if $T^k(x) + a_{k-1}T^{k-1}(x) + ... + a_1T(x) + a_0x = 0$ then

$$p_S(\lambda) = (-1)^k(\lambda^k + a_{k-1}\lambda^{k-1} + ... + a_1\lambda + a_0).$$

(iii) $p_S(\lambda)|p_T(\lambda)$, i.e., there exists a polynomial q such that $p_T = p_Sq$.

Proof

(i) Clearly, $T^j(X) \in W$, suppose that $\{x, T(x), ..., T^{k-1}(x)\}$ are linearly dependent, then

$$T^j = \sum_{l=1}^{j-1} c_l T^l x \in \text{span}(x, ..., T^{j-1}x),$$

some some $j < k$. Further,

$$T^{j+1} = \sum_{l=1}^{j-1} c_l T^{l+1} x \in \operatorname{span}(Tx, ..., T^j x) \subset \operatorname{span}(x, ..., T^{j-1} x).$$

Thus $W = \operatorname{span}(x, ..., T^{j-1}x)$ and $\dim W \leq j < k$. It shows that $\{x, T(x), ..., T^{k-1}(x)\}$ are linearly independent, thus they form a basis for W.
(ii) Consider the matrix of S in the basis $\{x, T(x), ..., T^{k-1}(x)\}$, it is

$$A_S = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 0 & 1 & \cdots & 0 & -a_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{k-1} \end{bmatrix}.$$

Since the characteristic polynomial of an operator does not depend on the choice of a basis, $p_S(\lambda) = p_A(\lambda) = \det(A - \lambda I)$. The cofactor expansion of the determinant across the last column gives

$$p_S(\lambda) = (-1)^k (a_0 + a_1\lambda + ... + a_{k-2}\lambda^{k-2} + (a_{k-1} + \lambda)\lambda^{k-1}).$$

(iii) There exists a basis for V of the form $\{x, Tx, ..., T^{k-1}x, y_1, ..., y_{n-k}\}$, where n is the dimension of V. Consider the blocks of the matrix of T in this basis,

$$A_T = [Tx, T^2x, ..., T^k x, Ty_1, ..., Ty_{n-k}] = \begin{bmatrix} A_S & C \\ 0 & B \end{bmatrix},$$

where A_S and B are square matrices of sizes $m \times m$ and $(n-m) \times (n-m)$ respectively. Then

$$\det(A_T - \lambda I) = \det(A_S - \lambda I) \det(B - \lambda I), \quad p_T(\lambda) = p_S(\lambda) p_B(\lambda).$$

℘ **The Cayley–Hamilton theorem**
Let T be a linear transformation of a finite dimensional space V and p_T be its characteristic polynomial then $p_T(T) = 0$ as an operator on V. In particular, a matrix satisfies its characteristic polynomial: $p_A(A) = 0$.

N.b. Since p_A is a polynomial of degree n, this implies that A^n can be replaced with a polynomial of degree at most $n - 1$. Hence $\exp(A)$ can be reduced to a polynomial in A of degree at most $n - 1$. This is the basis for the spectral decomposition below.

Proof
Let $x \in V$, W be the T-cyclic subspace generated by x and S be the restriction of T onto W. Then by the previous result $p_T(\lambda) = p_S(\lambda)q(\lambda) = q(\lambda)P_S(\lambda)$ as polynomials. Then $p_T(T) = q(T) \circ p_S(T)$ and $p_T(T)x = q(T)(p_S(T)x)$. But $p_S(\lambda) = (-1)^k(\lambda^k + a_{k-1}\lambda^{k-1} + ... + a_1\lambda + a_0)$

$$p_S(T)x = (-1)^k (T^k x + a_{k-1}T^{k-1}x + ... + a_1 Tx + a_0 x) = 0.$$

Thus $p_T(T)x = 0$ for any $x \in V$ and $p_T(T) = 0$ as an operator.

Ex.

- If D is a diagonal matrix $D = \text{diag}(\lambda_1, ...\lambda_n)$ then

$$p_D(D) = \text{diag}(p_D(\lambda_1), ..., p_D(\lambda_n)) = 0$$

since λ_j are roots of the characteristic polynomial.

- If A is a two-by-two matrix, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$p_A(\lambda) = A^2 - \text{tr}(A)\lambda + \det(A)$$

and

$$p_A(A) = \begin{bmatrix} a^2+bc & ab+bd \\ ca+dc & cb+d^2 \end{bmatrix} - (a+d)\begin{bmatrix} a & b \\ c & d \end{bmatrix} + (ad-bc)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Spectral decomposition

Generalized eigenspaces

Let $\lambda \in \mathbb{C}$ denote an eigenvalue of A. A vector $v \neq 0$ is called a **generalized eigenvector** if $(A - \lambda I)^k v = 0$ for some $k \in \mathbb{N}$; the set of all generalized eigenvalues is denoted $N(\lambda)$, then

$$N(\lambda) = \cup_k N_{k,\lambda}, \quad N_k := \ker\left((A-\lambda I)^k\right), \; E_\lambda = N_{1,\lambda} \subset N_{2,\lambda} \subset ... \subset N_{k,\lambda} \subset ... \subset N(\lambda),$$

each $N_{k,\lambda}$ is called a **generalized eigenspace** corresponding to the eigenvalue λ. Let

$$R_{k,\lambda} := \text{ran}((A-\lambda I)^k), \qquad k \in \mathbb{N}.$$

The vector spaces $R_{k,\lambda}$ satisfy

$$\{0\} \subset ... \subset R_{k+1,\lambda} \subset R_{k,\lambda} \subset ... \subset R_{1,\lambda} \subset \mathbb{C}^n.$$

The **Riesz index** m_λ is the minimal natural number that ends the chain $\{R_k\}_k$:

$$m_\lambda \overset{\text{def.}}{=} \min\{k \in \mathbb{N} \colon R_{k,\lambda} = R_{s,\lambda} \; \forall \; s \geq k\}.$$

$\wp$ Spectral decomposition theorem

Let A be an $n \times n$ matrix, $\{\lambda_1, ..., \lambda_m\}$ be its eigenvalues (without counting multiplicities) and $\lambda \in \{\lambda_1, ..., \lambda_m\}$. Then
(i) there exists a minimal integer, $k = k_\lambda \in \mathbb{N}$, such that $N(\lambda) = N_{k_\lambda,\lambda}$,
(ii) (Riesz decomposition) $k_\lambda = m_\lambda$, and

$$\mathbb{C}^n = N(\lambda) \oplus R(\lambda) = \ker\left((A-\lambda I)^{m_\lambda}\right) \oplus \text{ran}\left((A-\lambda I)^{m_\lambda}\right),$$

(iii) $\mathbb{C}^n$ can be decomposed into maximal generalized eigenspaces and $\dim(N\lambda_k) = \text{mult}(\lambda_k)$:

$$\mathbb{C}^n = \oplus_{k=1}^m N(\lambda_k).$$

Proof
(i) Since N_k are linear spaces, all contained in the n-dimensional space $\mathbb{C}^n$, and

$$\{0\} \subset N_k \subset N_{k+1} \subset \mathbb{C}^n, \qquad k \in \mathbb{N},$$

this chain must end:

$$\exists \text{ minimal } k_\lambda \geq 1; \qquad N_k = N_{k_\lambda} \quad \text{for all } k \geq k_\lambda.$$

(ii) The rank-nullity theorem implies that

$$\dim(N_{k,\lambda}) + \dim(R_{k,\lambda}) = n.$$

If $K_\lambda = N_{k_\lambda,\lambda}$ then

$$N_{k,\lambda} = N_{k_\lambda,\lambda} \quad \forall\, k \geq k_\lambda \;\Leftrightarrow\; \dim(N_{k,\lambda}) = \dim(N_{k_\lambda,\lambda}) \quad \forall\, k \geq k_\lambda$$
$$\Leftrightarrow\; \dim(R_{k,\lambda}) = \dim(R_{k_\lambda,\lambda}) \quad \forall\, k \geq k_\lambda \;\Leftrightarrow\; R_{k,\lambda} = R_{k_\lambda,\lambda} \quad \forall\, k \geq k_\lambda.$$

Hence $m_\lambda = k_\lambda$.

Suppose $B, C \in M_{n\times n}(\mathbb{C}^n)$. Then $BCx = 0 \iff Cx = 0$ means $\mathrm{ran}(C) \cap \ker(B) = \{0\}$. Thus $N_{k,\lambda} = N_{2k,\lambda}$ implies

$$\{0\} = \mathrm{ran}((A-\lambda I)^k) \cap \ker((A-\lambda I)^k) = R_{k,\lambda} \cap N_{k,\lambda}.$$

It holds for $k = k_\lambda = m_\lambda$ and by the rank-nullity theorem $\mathbb{C}^n = N(\lambda) \oplus R(\lambda)$.
(iii) Let $n_k := \mathrm{mult}(\lambda_k)$. Suppose that $n_k < m_{\lambda_k}$. Then there exists $x \in V$ such that

$$0 \neq y = (A-\lambda_k I)^{n_k} x, \quad (A-\lambda_k I)y = 0.$$

By the Caley-Hamilton theorem $p_A(A)x = q(A)(A-\lambda_k I)^{n_k}x = 0$. Then

$$0 = q(A)(A-\lambda_k I)^{n_k}x = q(A)y = q(\lambda_k)y \neq 0,$$

since $q(\lambda_k) \neq 0$. Therefore $n_k \geq m_{\lambda_k}$.

Now let $\mu \neq \lambda_1$ then $A - \mu I$ is injective on $N(\lambda_1)$. Suppose $(A-\mu I)x = 0$ then $Ax = \mu x$ and $(A-\lambda_k I)^j x = (\mu-\lambda_1)^j x \neq 0$ when $x \neq 0$. Thus $A - \mu I$ is invertible on $N(\lambda_1)$. The characteristic polynomial of A has the following factorization

$$p_A(\lambda) = (\lambda-\lambda_1)^{n_1} \prod j = 2^m(\lambda-\lambda_j)^{n_j} = (\lambda-\lambda_1)^{n_1} q(\lambda).$$

Then $p_A(A) = (A-\lambda_1 I)^{n_1} q(A)$ and the restriction of $q(A)$ on $N(\lambda_1) = \ker(A-\lambda_1 I)^{n_1}$ is invertible. Thus $\ker(q(A)) \cap \ker(A-\lambda_1 I)^{n_1} = \{0\}$. But the Caley-Hamilton theorem implies that $\mathrm{ran}(q(A)) \subset \ker(A-\lambda_1 I)^{n_1}$ and by the rank-nullity theorem

$$\dim(N(\lambda_1)) + \dim(\ker(q(A))) = \dim(\ker(A-\lambda_1 I)^{n_1}) + \dim(\ker(q(A))) \geq$$
$$\dim(\mathrm{ran}(q(A))) + \dim(\ker(q(A))) = n$$

Thus $C^n = N(\lambda_1) \oplus \ker(q(A))$. Similarly, $\ker(q(A)) = N(\lambda_2) \oplus \ker(q_1(A))$. Then

$$\mathbb{C}^n = \oplus_{k=1}^m \ker((A-\lambda_k I)^{n_k}) = \oplus_{k=1}^m N(\lambda_k).$$

Note that $N(\lambda_k)$ is an invariant subspace of A and A restricted to $N(\lambda_k)$ has only one eigenvalue λ_k. Now if we choose a basis for $\mathbb{C}^n$ which is the union of bases for each $N(\lambda_k)$ then we get a block diagonal matrix similar to A for which the characteristic polynomial equals

$$\prod_{k=1}^m (\lambda-\lambda_k)^{d_k}$$

where $d_k = \dim(N(\lambda_k))$. Thus $d_k = m_{\lambda_k}$.

The matrix form of the spectral decomposition and exponential of a matrix
According to the above, $\mathbb{C}^n = \oplus_{k=1}^m N(\lambda_k)$ has a basis of generalized eigenvectors. Let

$$A_k := A|_{N(\lambda_k)}, \qquad I_k := I|_{N(\lambda_k)}, \qquad \tilde{N}_k := A_k - \lambda_k I_k, \quad k = 1, \ldots, m,$$

be the restrictions of the mappings A, I and $A - \lambda_k I$ onto the eigenspaces $N(\lambda_k)$ (meaning that they act only on the basis vectors of the corresponding eigenspaces). Then $\tilde{N}_k$ is nilpotent, since

$$\tilde{N}_k^{m_{\lambda_k}} = 0$$

on the generalized eigenspace $N(\lambda_k)$ (this is the definition of m_{λ_k}). *In our basis of generalized eigenvectors* A takes the form

$$\begin{bmatrix} [A_1] & 0 & 0 & \dots & 0 \\ 0 & [A_2] & 0 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & \dots & 0 & [A_m] \end{bmatrix}_{n\times n} = \begin{bmatrix} [\lambda_1 I_1 + \tilde{N}_1] & 0 & 0 & \dots & 0 \\ 0 & [\lambda_2 I_2 + \tilde{N}_2] & 0 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \dots & & 0 & [\lambda_m I_m + \tilde{N}_m] \end{bmatrix}_{n\times n}.$$

Because $\tilde{N}_k I_k = I_k \tilde{N}_k$, $\exp(t\lambda_k I_k) = e^{t\lambda} I_k$, and $\tilde{N}_k^{m_{\lambda_k}} = 0$ one has

$$\exp(tA_k) = \exp\big(t(\lambda_k I_k + \tilde{N}_k)\big) = \exp\big(t\lambda_k I_k\big)\exp\Big(t\tilde{N}_k\Big) = e^{t\lambda_k}\big(I_k + t\tilde{N}_k + \ldots + \frac{(t\tilde{N}_k)^{m_{\lambda_k}-1}}{(m_{\lambda_k}-1)!}\big),$$

and then

$$\exp(t \underbrace{T[A_k]_k T^{-1}}_{tA \text{ in original basis}}) = T\exp(t[A_k]_k)T^{-1} \qquad (T \text{ change-of-basis matrix}).$$

One only needs to find suitable bases for $N(\lambda_k)$, $k = 1, \ldots, m$.

Ex.

The matrix

$$A := \begin{bmatrix} 0 & -8 & 4 \\ 0 & 2 & 0 \\ 2 & 3 & -2 \end{bmatrix}$$

has eigenvalues $\lambda_{1,2} = 2$ and $\lambda_3 = -4$. Its generalized eigenvectors solve

$$(A-2I)^2 v = \begin{bmatrix} 12 & 28 & -24 \\ 0 & 0 & 0 \\ -12 & -28 & 24 \end{bmatrix} v = 0 \quad \Leftrightarrow \quad v = s\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 0 \\ 6 \\ 7 \end{bmatrix} \qquad s,t \in \mathbb{C},$$

and

$$(A+4I)\,v = 0 \quad \Leftrightarrow \quad v = s\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad s \in \mathbb{C}.$$

Let

$$T := \begin{bmatrix} 2 & 0 & -1 \\ 0 & 6 & 0 \\ 1 & 7 & 1 \end{bmatrix} \quad \text{so that} \quad T^{-1} = \frac{1}{18}\begin{bmatrix} 6 & -7 & 6 \\ 0 & 3 & 0 \\ -6 & -14 & 12 \end{bmatrix}, \quad T^{-1}AT = \begin{bmatrix} 2 & -10 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -4 \end{bmatrix}.$$

In the basis given by T we have

$$I_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \tilde{N}_1 = \begin{bmatrix} 0 & -10 \\ 0 & 0 \end{bmatrix} \quad \text{with} \quad A_1 = 2I_1 + \tilde{N}_1,$$

$$\exp(tA_1) = \exp(2tI_1)\exp(t\tilde{N}_1) = e^{2t}(I_1 + t\tilde{N}_1) = e^{2t}\begin{bmatrix} 1 & -10t \\ 0 & 1 \end{bmatrix},$$

and

$$\exp(tT^{-1}AT) = \begin{bmatrix} e^{2t} & -10te^{2t} & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{-4t} \end{bmatrix}.$$

Expressed in the original basis,

$$\exp(tA) = T\exp(tT^{-1}AT)T^{-1} = \frac{1}{9}e^{2t}\begin{bmatrix}6 & -7 & 6\\ 0 & 9 & 0\\ 3 & 7 & 3\end{bmatrix} - \frac{1}{3}te^{2t}\begin{bmatrix}0 & 10 & 0\\ 0 & 0 & 0\\ 0 & 5 & 0\end{bmatrix} + \frac{1}{9}e^{-4t}\begin{bmatrix}3 & 7 & -6\\ 0 & 0 & 0\\ -3 & -7 & 6\end{bmatrix}.$$

The Jordan normal form

The Jordan normal form corresponds to a spectral decomposition in which the bases for $N(\lambda_k)$ are chosen such that the nilpotent matrices $\tilde{N}_k$ have the special form

$$\tilde{N}_{\lambda_k} = \begin{bmatrix}0 & j_1 & 0 & \dots & 0\\ 0 & 0 & j_2 & \ddots & \vdots\\ \vdots & & \ddots & \ddots & 0\\ 0 & 0 & \dots & 0 & j_{n_k-1}\\ 0 & 0 & \dots & 0 & 0\end{bmatrix}, \qquad j_l \in \{0,1\}, \quad l = 1,\dots,n_k-1,$$

with $n_k = \text{mult}(\lambda_k)$, and

$$A_k = \lambda_k I_k + \tilde{N}_{\lambda_k} = \begin{bmatrix}\lambda_k & j_1 & 0 & \dots & 0\\ 0 & \lambda_k & j_2 & \ddots & \vdots\\ \vdots & & \ddots & \ddots & 0\\ 0 & 0 & \dots & \lambda_k & j_{n_k-1}\\ 0 & 0 & \dots & 0 & \lambda_k\end{bmatrix}.$$

To obtain this, given an eigenvalue λ, pick a generalized eigenvector

$$v_{m_\lambda} \in \ker(A-\lambda I)^{m_\lambda}, \quad v_{m_\lambda} \notin \ker(A-\lambda I)^{m_\lambda-1}$$

and set

$$v_{m_\lambda-1} := (A-\lambda I)v_{m_\lambda}, \quad \dots \quad , v_1 := (A-\lambda I)^{m_\lambda-1}v_{m_\lambda},$$

so that

$$v_j \in \ker\left((A-\lambda I)^j\right), \quad v_j \notin \ker\left((A-\lambda I)^{j-1}\right), \qquad j = 1,\dots,m_\lambda.$$

The **Jordan chain** $\{v_1,\dots,v_{m_\lambda}\}$ is a basis for a subspace of $N(\lambda)$, on which

$$\tilde{N}v_j = (A-\lambda I)v_j = v_{j-1}, \qquad j = 1,\dots,m_\lambda,$$

if we let $v_0 := 0$. Hence, the j:th column of $\tilde{N}$ is v_{j-1}. This gives the nilpotent part of a so-called **Jordan block** (with ones above the diagonal, all other elements zero). If $m_\lambda < n_k$ additional Jordan chains need to be added. Each chain gives rise to a Jordan block; adding the different chains into a basis for $N(\lambda)$ gives the form of $\tilde{N}$ above.

Ex. (continued from above)

The eigenvalues of

$$A := \begin{bmatrix}0 & -8 & 4\\ 0 & 2 & 0\\ 2 & 3 & -2\end{bmatrix}$$

are $\lambda_1 = 2$ (double) and $\lambda_2 = -4$ (simple).

Since $\text{mult}(2) = 2$, we have $k_2 = m_2 \le 2$, so we can start the Jordan chain by looking for a vector in N_2 (which equals the maximal generalized eigenspace $N(2)$). Candidates are (cf.

above):

$$u = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad w = \begin{bmatrix} 0 \\ 6 \\ 7 \end{bmatrix}.$$

Since $(A - 2I)u = 0$ we have $u \in N_1$, whereas

$$(A - 2I)w = \begin{bmatrix} -20 \\ 0 \\ -10 \end{bmatrix} = -10u$$

implies that

$$v_2 := w \in N_2 \setminus N_1 \quad \text{whereas} \quad v_1 := (A - 2I)w = -10u \in N_1.$$

The Jordan block corresponding to the simple eigenvalue -4 consists of just the eigenvalue itself, and the eigenvector spanning the one-dimensional eigenspace $N(-4)$ is $\tilde{v}_1 := (-1, 0, 1)$, as calculated above.

The change-of-basis matrix is thus given by

$$T := [v_1 \; v_2 \; \tilde{v}_1] = \begin{bmatrix} -20 & 0 & -1 \\ 0 & 6 & 0 \\ -10 & 7 & 1 \end{bmatrix},$$

in which the linear transformation expressed by A in the original basis takes the Jordan normal form[1]

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -4 \end{bmatrix}.$$

℘ **The spectral theorem for finite dimensional self-adjoint operators**
Let $A \in M_{n\times n}(\mathbb{K})$ be symmetric (hermitian). Then there exists an orthonormal basis $\{Q_j\}_{j=1}^n$ for $\mathbb{R}^n$ ($\mathbb{C}^n$) of eigenvectors of A, such that

$$A = QDQ^*,$$

where Q is the orthogonal (unitary) matrix with columns $(Q_j)_j$ and D is a diagonal matrix with the eigenvalues of A as its diagonal elements.

N.b. It is possible to extend the spectral theorem to all **normal** matrices, characterized by $AA^* = A^*A$.[2]

Proof
Each eigenspace is maximal: Let λ an eigenvalue of A and pick $x \in \ker((A-\lambda I)^2)$. Since A is self-adjoint with real eigenvalues we have

$$0 = \langle (A - \lambda I)^2 x, x\rangle = \|(A - \lambda I)x\|^2 \quad \Longrightarrow \quad x \in \ker(A - \lambda I).$$

Hence the Riesz index of λ is 1, and all eigenvalues are semi-simple, meaning that $\dim(\ker(A-\lambda I)) = \text{mult}(\lambda)$.

Applying the spectral decomposition: The statement now follows from the spectral (or Jordan) decomposition: The maximal generalized eigenspaces coincide with the eigenspaces,

[1] As can be seen, the spectral decomposition above brought as very close to the Jordan normal form, which will typically happen if the Jordan chains are few or short (low algebraic multiplicity).
[2] In fact, the class of normal matrices is the biggest class of matrices for which the spectral theorem holds.

these are mutually orthogonal, and we may pick an orthonormal basis for each of them. Together, these form an orthonormal basis for K^n, described by Q.

Ex. Let A_n be an $n \times n$ matrix with $a_{jj+1} = a_{j+1j} = 1/2$ and all other entries that are zeros. This matrix has the following eigenvectors and eigenvalues

$$v_k = \{\sin kj\pi/(n+1)\}_{j=1}^n, \ \lambda_k = \cos k\pi/(n+1), \quad k = 1, ..., n.$$

Thus it is similar to the diagonal matrix $D = \text{diag}(\cos k\pi/(n+1))$.
The characteristic polynomials p_n of A_n satisfy

$$p_1(\lambda) = -\lambda, \ p_2(\lambda) = \lambda^2 - 1/4, \ p_{n+1}(\lambda) = -\lambda p_n(\lambda) - 1/4 p_{n-1}(\lambda).$$

Positive definiteness and the singular value decomposition

Let $A = A^t \in M_{n\times n}(\mathbb{R})$ be a symmetric matrix.

- A is said to be **positive definite** if

$$\langle Ax, x\rangle = x^t Ax > 0 \quad \text{for} \quad x \neq 0.$$

- A is said to be **positive semi-definite** if $\langle Ax, x\rangle = x^t Ax \geq 0$ for all $x \in \mathbb{R}^n$.

Ex.

- $A = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$ is positive definite since

$$\langle Ax, x\rangle = 2(x_1^2 - x_1x_2 + x_2^2) = x_1^2 + (x_1 - x_2)^2 + x_2^2 > 0 \quad \text{when } x \neq 0.$$

- Let A be an 3×3 matrix with $a_{jj} = 2$ and $a_{ij} = -1$ for $i \neq j$ then

$$\langle Ax, x\rangle = 2(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3 - x_3x_1) = (x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2 \geq 0$$

  This matrix is positive semi-definite but not positive definite.

Characterization of positive definite matrices

A symmetric matrix $A = A^t \in M_{n\times n}(\mathbb{R})$ is positive definite exactly if one (and hence all) of the following conditions hold:

- $\langle A\cdot, \cdot\rangle$ defines an inner product.
- All the eigenvalues of A are strictly positive.
- $A = R^tR$ for some invertible matrix R.

Proof (contains important methods)

Inner-product property. Assume that A is positive definite. Since $x \mapsto Ax$, $\mathbb{R}^n \to \mathbb{R}^n$, is linear, and the usual inner product is sesqui-linear (linear in its first argument, anti-linear in its second), the form

$$(x, y) \mapsto \langle Ax, y\rangle, \qquad \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R},$$

is also sesqui-linear. Furthermore,

$$\langle Ax, y\rangle = \langle x, Ay\rangle = \langle Ay, x\rangle,$$

by the symmetry of A and of the standard inner product, and

$$\langle Ax, x\rangle > 0 \quad \text{for} \quad x \neq 0,$$

by the definition of positive definiteness, so the inner product $\langle A\cdot, \cdot\rangle$ is non-degenerate symmetric.

Considering the same arguments, one also sees that $\langle A\cdot, \cdot\rangle$ cannot be an inner product unless A is positive definite.

Eigenvalue property: Since A is symmetric, there exists an orthonormal basis of eigenvectors $\{v_1, \dots, v_n\}$ with $Av_j = \lambda_j v_j$. Let x_j be the coordinates of a vector x in this basis. Then

$$\langle Ax, x\rangle = \Big\langle A\sum_{j=1}^{n} x_j v_j, \sum_{k=1}^{n} x_k v_k\Big\rangle = \Big\langle \sum_{j=1}^{n} x_j (Av_j), \sum_{k=1}^{n} x_k v_k\Big\rangle = \sum_{j,k=1}^{n} \lambda_j \langle x_j v_j, x_k v_k\rangle = \lambda_j x_j^2 > 0$$

for all $x \neq 0$ if and only if $\lambda_j > 0$ for all $j = 1, \dots, n$.

Factorization property: If $A = R^t R$ with R invertible, then

$$\langle Ax, x\rangle = \langle R^t Rx, x\rangle = \|Rx\|^2 > 0,$$

unless $Rx = 0$, which happens only if $x = 0$ (as R is invertible).

Contrariwise, if A is symmetric and positive definite, we can write

$$A = QDQ^t = Q\sqrt{D}\sqrt{D}Q^t = Q(\sqrt{D})^t\sqrt{D}Q^t = (\sqrt{D}Q^t)^t(\sqrt{D}Q^t),$$

where Q is an orthogonal matrix of eigenvectors, $D = (\lambda_j)_j$ is a diagonal matrix with positive eigenvalues, and $\sqrt{D} = (\sqrt{\lambda_j})_j$ has $\sqrt{\lambda_j}$ as diagonal elements. Thus, $A = R^t R$.

Ex. Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ then

$$p_A(\lambda) = -\lambda^3 + 6\lambda^2 - 9\lambda + 4 = -(\lambda - 1)^2(\lambda - 4).$$

The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 4$, they are positive, the matrix is positive definite. An orthonormal basis of eigenvectors is, for example,

$$v_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad v_2 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \quad v_3 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Then

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix},$$

and $A = R^t R$, where $R = \sqrt{D}Q^t$,

$$R = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \\ \frac{2}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{2}{\sqrt{3}} \end{bmatrix}$$

The Cholesky decomposition

An alternative to the decomposition $A = R^tR$ obtained above for positive definite matrices is the **Cholesky decomposition**: it gives a representation of a positive definite matrix in the form $A = R^tR$, where R is upper triangular. If one divides out the main pivots (diagonal elements) in the LU-factorization of A, one gets an LDU-**decomposition**, D being a diagonal matrix with the main pivots as diagonal elements, and L *and* U *having only unit elements on their main diagonals.* This factorization is unique. We thus obtain

$$LDU = A = A^t = (LDU)^t = U^tD^tL^t = U^tDL^t,$$

where U^t is lower triangular, and L^t is upper triangular. By uniqueness of the LDU-factorization, we must have $L = U^t$ and $U = L^t$. Consequently,

$$A = LDL^t = L\sqrt{D}\sqrt{D}L^t = L\sqrt{D}(L\sqrt{D})^t,$$

with $L\sqrt{D}$ being invertible ($\sqrt{D}$ has only positive elements along the diagonal and L is lower triangular with units along the diagonal).

Ex. Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ as above, Then LDU-decomposition of A is

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/2 & 1/3 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 4/3 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 1/2 & 1/2 \\ 0 & 1 & 1/3 \\ 0 & 0 & 1 \end{bmatrix},$$

and $A = R^tR$, where now $R = \sqrt{D}U = (L\sqrt{D})^t$,

$$R = \begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{\sqrt{3}}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{2}{\sqrt{3}} \end{bmatrix}$$

The singular value theorem

Let $A \in M_{m\times n}(\mathbb{R})$ be the realization of a linear transformation $\mathbb{R}^n \to \mathbb{R}^m$, such that $\text{rank}(A) = r$. Then A^tA is a positive semi-definite matrix of rank r, and there exists an orthonormal basis of eigenvectors of A^tA,

$$\{v_1, \ldots, v_n\} \subset \mathbb{R}^n, \quad \text{with eigenvalues} \quad \sigma_1^2 \geq \ldots \geq \sigma_r^2 > 0, \quad \sigma_{r+1}^2 = \ldots = \sigma_n^2 = 0,$$

and a corresponding orthonormal basis

$$\{u_1, \ldots, u_r, u_{r+1}, \ldots, u_m\} := \{\frac{1}{\sigma_1}Av_1, \ldots, \frac{1}{\sigma_r}Av_r, u_{r+1}, \ldots, u_m\}$$

for $\mathbb{R}^m$ ($u_{r+1}, \ldots, u_m$ arbitrary to fit the orthonormal basis), such that

$$Av_j = \begin{cases} \sigma_j u_j, & j = 1, \ldots, r, \\ 0, & j = r+1, \ldots, n. \end{cases}$$

The unique scalars $\sigma_1, \ldots, \sigma_r, 0, \ldots, 0$ (extended to a total of $\min(m, n)$) are called **singular values** of A. If

$$(\Sigma_{ij})_{ij} := (\delta_{ij}\sigma_j)_{ij} \in M_{m\times n}(\mathbb{R}), \quad \sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix},$$

is the diagonal matrix with $\sigma_1, \ldots, \sigma_r, 0, \ldots, 0$ on its main diagonal, $U = (u_j)_j \in M_{m\times m}(\mathbb{R})$ is the orthogonal matrix with u_j as columns, and $V = (v_j)_j \in M_{n\times n}(\mathbb{R})$ is the orthogonal matrix with v_j as columns, it follows that

$$A = U\Sigma V^{-1} = U\Sigma V^t.$$

This is the **singular value decomposition** of A.

N.b. The singular values for A^t equals those of A. For A^t, the orthonormal bases V and U are simply exchanged (in comparison to A).

Ex. Find the singular value decomposition of

$$A = \begin{bmatrix} 0 & 3 & 6 \\ 4 & 5 & 2 \end{bmatrix}.$$

Let start with A^t then

$$AA^t = \begin{bmatrix} 45 & 27 \\ 27 & 45 \end{bmatrix},$$

the eigenvalues are $\lambda_1 = 72$ and $\lambda_2 = 18$. The corresponding eigenvectors are

$$v_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \quad A^t v_1 = \begin{bmatrix} 2\sqrt{2} \\ 4\sqrt{2} \\ 4\sqrt{2} \end{bmatrix}, \quad A^t v_2 = \begin{bmatrix} 2\sqrt{2} \\ \sqrt{2} \\ -2\sqrt{2} \end{bmatrix}.$$

Normalizing and adding the third orthogonal vector we obtain the singular value decomposition of A:

$$A = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 6\sqrt{2} & 0 & 0 \\ 0 & 3\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$$

The pseudoinverse

A finite-dimensional linear transformation can always be inverted on its range. Let $A \in M_{m\times n}(\mathbb{R})$ be the realization of a linear transformation $\mathbb{R}^n \to \mathbb{R}^m$, and let

$$A|_{\ker(A)^\perp} : \ker(A)^\perp \to \operatorname{ran}(A)$$

be the restriction of A to the orthogonal complement of its null space. Then (according to the rank–nullity theorem) $A|_{\ker(A)}$ is bijective. Its inverse, $A^\dagger$, is called the **(Moore-Penrose) pseudoinverse**:

$$A^\dagger : \operatorname{ran}(A) \to \ker(A)^\perp, \qquad A^\dagger(Ax) = x, \text{ for } x \in \ker(A)^\perp = \operatorname{ran}(A^t).$$

The pseudoinverse from the singular value decomposition

If $A = U\Sigma V^t$ is the singular value decomposition of $A \in M_{m\times n}(\mathbb{R})$, using the bases $\{u_j\}_j$ and $\{v_j\}_j$, we only have to invert $Av_j = \sigma_j u_j$ for $j = 1, \ldots, r$.

More precisely, if

$$(\Sigma^\dagger_{ij})_{ij} = (\delta_{ij}\frac{1}{\sigma_j})_{ij} \in M_{n\times m}(\mathbb{R})$$

is the diagonal matrix with the reciprocals of the singular values along its main diagonal, then

$$A^\dagger = V\Sigma^\dagger U^t$$

is the singular value decomposition of the pseudoinverse $A^\dagger$.

Ex. Find the pseudoinverse of

$$A = \begin{bmatrix} 0 & 3 & 6 \\ 4 & 5 & 2 \end{bmatrix}.$$

From the singular value decomposition of A we have

$$A = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 6\sqrt{2} & 0 \\ 0 & 3\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \end{bmatrix}.$$

Then

$$A^\dagger = \begin{bmatrix} 1/3 & 2/3 \\ 2/3 & 1/3 \\ 2/3 & -2/3 \end{bmatrix} \begin{bmatrix} 1/6\sqrt{2} & 0 \\ 0 & 1/3\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} -1/12 & 5/36 \\ 0 & 1/9 \\ 1/6 & -1/18 \end{bmatrix}.$$

$$A^\dagger A = \begin{bmatrix} 5/9 & 4/9 & -2/9 \\ 4/9 & 5/9 & 2/9 \\ -2/9 & 2/9 & 8/9 \end{bmatrix}$$