## TMA 4230 – PROBLEM SET 1, 2014

**Problem 1:** Let $(X, d_X)$ and $(Y, d_Y)$ be two metric spaces. Then a function $f : X \to Y$ is called a homeomorphism if it is a continuous bijection with a continuous inverse $f^{-1}$. In this case one says that $(X, d_X)$ and $(Y, d_Y)$ are homeomorphic.

(a) Suppose $a, b$ are finite real numbers. Show that $(0, 1)$ is homeomorphic to $(a, b)$.

(b) Show that $\mathbb{R}$ is homeomorphic to $(0, 1)$.

(c) Give an example that shows that completeness is not preserved under a homeomorphism.

**Problem 2:** Show that a normed space $(V, \|.\|)$ is complete if and only if $\sum_{j=1}^{\infty} f_j$ converges in norm whenever $\sum_{j=1}^{\infty} \|f_j\|$ converges.

**Problem 3:** The **Cantor set** $\mathcal{C}$ is defined as $\mathcal{C} = \int_{n=1}^{\infty}$, where $I_{n+1}$ is constructed by trisecting $I_n$ and removing the middle third, $I_0$ being the closed interval $[0, 1]$. Show that $\mathcal{C}$ has the following properties:

(a) $\mathcal{C}$ is non-empty and uncountable.

(b) $\mathcal{C}$ is closed and nowhere dense.

**Problem 4:** Let $X$ be the normed space $(C[0, 1], \|.\|_1)$, where $C[0, 1]$ is the set of all real-valued continuous functions on $[0, 1]$ and $\|f\|_1 = \int_0^1 |f(t)|dt$. Show that $(C[0, 1], \|.\|_1)$ is not complete.