TMA4230 FUNCTIONAL ANALYSIS, WEEK 7

LAST WEEK: Last week we looked at closed linear operators and the closed graph theorem and I started on a short introduction to topology with applications to functional analysis in mind.

- Closed linear operators and the closed graph theorem (Section 4.13).
- Page 32–39 of the notes.

THIS WEEK: This week we first finish the introduction to topology by looking at compactness and Tychonoff's theorem, and we will then use that to prove the Banach-Alaoglu Theorem. If time permits we will also look at the Stone-Weierstrass Theorem which is a generalization of the Weierstrass Theorem.

- Compactness and Tychonoff's theorem (page 39–43 of the notes).
- The Banach-Alaoglu Theorem (page 52–54 of the notes).
- The Stone-Weierstrass Theorem (extra notes which you can get in class or by sending me an email).

NEXT WEEK: Next week we will first look at the Stone-Weierstrass Theorem if we haven't already done that. Then we will begin on spectral theory. We will do things in another order than the book does and instead follow the lead of the notes, but in the end we will cover all of Chapter 7 and page 61–65 of the notes. We begin next week with some basic concepts in spectral theory and spectral theory in finite dimension and then move on to spectral theory for Banach algebras.

- Spectral theory in finite dimensions (Section 7.1).
- Some basic concepts in spectral theory (Section 7.2).
- Spectral theory for Banach algebras (Section 7.6 and page 61–65 of the notes).
- Further properties of resolvent and spectrum (Section 7.4).

EXERCISES FOR NEXT WEEK: The following 4 exercises.

EXERCISE 1: Let Y be a dense subset of a topological space X. Show that $\overline{Y \cap A} = \overline{A}$ for every open subset A of X.

EXERCISE 2: Let X and Y be topological spaces and consider the product space $X \times Y$ equipped with the product topology. Show that if $A \subseteq X$ and $B \subseteq Y$, then $\overline{A \times B} = \overline{A} \times \overline{B}$ and $(A \times B)^\circ = A^\circ \times B^\circ$ (here A° denotes the interior of A, i.e. $A^\circ = \{x \in A \mid \exists U \subseteq A,\ x \in U,\ U \text{ is open}\}$).

EXERCISE 3: Let X and Y be topological spaces and f a map from X to Y. Show that the graph

$$\mathscr{G}(f) = \{(x,y) \in X \times Y \mid f(x) = y\}$$

is closed in $X \times Y$ if Y is Hausdorff and f is continuous.

Show then that if X and Y both are compact and Hausdorff and $\mathscr{G}(f)$ is closed in $X \times Y$, then f is continuous.

EXERCISE 4: Let $(f_i)_{i \in I}$ be a net of real continuous functions on a compact space X. Assume that $i \leq j$ implies that $f_i(x) \leq f_j(x)$ for every $x \in X$ and that there is a continuous function f on X such that $\lim_{i \in I} f_i(x) = f(x)$ for every $x \in X$. Show that $(f_i)_{i \in I}$ converges uniformly to f, i.e. $||f - f_i||_\infty \to 0$.