Department of Mathematical Sciences
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TMA4230 FUNCTIONAL ANALYSIS, WEEK 4

LAST WEEK: Last week we used the Hahn-Banach Theorem to study *bounded linear functionals on* $C[a,b]$ and introduce *the Riemann-Stieltjes integral* (Section 4.4). Something we will need later for the spectral theorem. We then looked at Riesz's representations theorems for Hilbert spaces (Section 3.8) and the adjoint of a bounded operator between Hilbert spaces (Section 3.9).

- The Riemann-Stieltjes integral (Section 4.4).
- Riesz's representations theorems for Hilbert spaces (Section 3.8).
- The adjoint of a bounded operator between Hilbert spaces (Section 3.9).

THIS WEEK: This week we will first look at adjoint operators between normed spaces and see how this is related to the adjoint of a bounded operator between Hilbert spaces (Section 4.5). We will then look at reflexive spaces and, among other things, show that every Hilbert space is reflexive (Section 4.6). We will look at Baire category theorem, the uniform boundedness theorem (Section 4.7) and, if times permit it, strong and weak convergence (Section 4.8).

- Adjoint operators (Section 4.5).
- Reflexive spaces (Section 4.6).
- The Baire category theorem and the uniform boundedness theorem (Section 4.7).
- Strong and weak convergence (Section 4.8).

NEXT WEEK: If we don't manage to finish Section 4.8 this week, we will do it next week. We will then look at convergence of sequences of operators and functionals (Section 4.9), and then the open mapping theorem (section 4.12) and closed linear operators and the closed graph theorem (section 4.13).

- Strong and weak convergence (Section 4.8).
- Convergence of sequences of operators and functionals (Section 4.9).
- The open mapping theorem (Section 4.12).
- Closed linear operators and the closed graph theorem (Section 4.13).

EXERCISES FOR NEXT WEEK: 4.5.2, 4.5.9, 4.5.10, 4.6.4 and 4.6.7 plus the following two exercises.

EXERCISE 1: Consider the Banach spaces $c_0 = \{(x_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} \mid \lim_{n\to\infty} x_n = 0\}$, $l^1 = \{(x_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} \mid \sum_{n=1}^{\infty} |x_n| < \infty\}$ and $l^\infty = \{(x_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} \mid \sup_{n\in\mathbb{N}} |x_n| < \infty\}$ where we use the norm $||(x_n)_{n\in\mathbb{N}}||_\infty = \sup_{n\in\mathbb{N}} |x_n|$ on c_0 and l^∞, and the norm $||(x_n)_{n\in\mathbb{N}}||_1 = \sum_{n=1}^{\infty} |x_n|$ on l^1. Remember that $(c_0)'$ is isomorphic to l^1 and that $(l^1)'$ is isomorphic to l^∞.

Define $T : l^1 \to c_0$ by $(Tx)_n = \sum_{m=n}^{\infty} x_m$ for $n \in \mathbb{N}$ and $(x_m)_{m\in\mathbb{N}} \in l^1$. Show that $T \in B(l^1, c_0)$ and give an explicit formula for $T^\times$ in $B(l^1, l^\infty)$.

EXERCISE 2: Let $1 < p < \infty$. Show that the Banach space $l^p = \{(x_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} \mid \sum_{n=1}^{\infty} |x_n|^p < \infty\}$ with the norm $||(x_n)_{n\in\mathbb{N}}||_p = (\sum_{n=1}^{\infty} |x_n|^p)^{1/p}$ is reflexive. (Hint: Example 2.10.7).