TMA4230 FUNCTIONAL ANALYSIS, WEEK 3

LAST WEEK: Last week we looked at several versions of the Hahn-Banach

- Hahn-Banach Theorem (Section 4.2–4.3, and (a modification of) Corollary 60 from the notes).

THIS WEEK: This week we will use the Hahn-Banach Theorem to study *bounded linear functionals on* $C[a,b]$ and introduce *the Riemann-Stieltjes integral* (Section 4.4). Something we will need later for the spectral theorem. We will then move on to *adjoint operators* (Section 4.5) and *reflexive spaces* (Section 4.6). We will also make a brief review of Riesz representation theorem for Hilbert spaces (Section 3.8) and adjoint operators on Hilbert spaces (Section 3.9).

- The Riemann-Stieltjes integral (Section 4.4).
- Riesz representation theorem for Hilbert spaces (Section 3.8).
- Adjoint operators on Hilbert spaces (Section 3.9).
- Adjoint operators (Section 4.5).
- Reflexive spaces (Section 4.6).

NEXT WEEK: Next week we will look at Baire category theorem, the uniform boundedness theorem, strong and weak convergence, and convergence of sequences of operators and functionals.

- The Baire category theorem and the uniform boundedness theorem (Section 4.7).
- Strong and weak convergence (Section 4.8).
- Convergence of sequences of operators and functionals (Section 4.9).

EXERCISES FOR NEXT WEEK: 4.5.2, 4.5.9, 4.5.10, 4.6.4 and 4.6.7 plus the following two exercises.

EXERCISE 1: Consider the Banach spaces $c_0 = \{(x_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} \mid \lim_{n\to\infty} x_n = 0\}$, $l^1 = \{(x_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} \mid \sum_{n=1}^{\infty} |x_n| < \infty\}$ and $l^\infty = \{(x_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} \mid \sup_{n\in\mathbb{N}} |x_n| < \infty\}$ where we use the norm $||(x_n)_{n\in\mathbb{N}}||_\infty = \sup_{n\in\mathbb{N}} |x_n|$ on c_0 and l^∞, and the norm $||(x_n)_{n\in\mathbb{N}}||_1 = \sum_{n=1}^{\infty} |x_n|$ on l^1. Remember that $(c_0)'$ is isomorphic to l^1 and that $(l^1)'$ is isomorphic to l^∞.

Define $T : l^1 \to c_0$ by $(Tx)_n = \sum_{m=n}^{\infty} x_m$ for $n \in \mathbb{N}$ and $(x_m)_{m\in\mathbb{N}} \in l^1$. Show that $T \in B(l^1, c_0)$ and give an explicit formula for $T^\times$ in $B(l^1, l^\infty)$.

EXERCISE 2: Let $1 < p < \infty$. Show that the Banach space $l^p = \{(x_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} \mid \sum_{n=1}^{\infty} |x_n|^p < \infty\}$ with the norm $||(x_n)_{n\in\mathbb{N}}||_p = (\sum_{n=1}^{\infty} |x_n|^p)^{1/p}$ is reflexive. (Hint: Example 2.10.7).