Let \( \lambda \in \mathbb{C} \) and \( \lambda_1, \ldots, \lambda_k \) the roots of the polynomial \( p(x) \), i.e., \( p(x) - \lambda = \alpha(x - \lambda_1) \cdots (x - \lambda_k) \), so that \( p(\lambda) = 0 \) is inscribed if and only if \( \lambda - \lambda_i \) is inscribed for all \( i \).

That is \( \lambda \in \sigma(p(\lambda)) \Longleftrightarrow \lambda \in \sigma(A) \) for all \( i \) \( \Rightarrow \lambda \notin p(\sigma(A)) \)

(since \( p(\lambda) = 0 \) for all \( i \)).

7.1.15

\[
\begin{align*}
\mathbb{X} & = \{ \text{polynomials } x(t) = a_0 + a_1 t + \ldots + a_m t^{m-1} \} \\
T : \mathbb{X} & \rightarrow \mathbb{X}, \quad (Tx)(t) = x'(t) = a_1 t + 2a_2 t^2 + \ldots + (m-1)a_{m-1} t^{m-2}
\end{align*}
\]

Clearly \( (Tx)(t) = x'(t) \Rightarrow \lambda x(t) \quad \Rightarrow \lambda = 0 \) and \( x(t) = a_0 \) for some constant.

Hence 0 is an eigenvalue of algebraic multiplicity \( m \), and geometric multiplicity 1 (the eigenspace \( \mathcal{E}_0 \) corresponding to \( \lambda = 0 \) is 1-dimensional, consisting of the constant functions).

7.2.3

Suppose \( \lambda \) is an eigenvalue of \( T \) and \( \mathcal{E}_\lambda \) the corresponding eigenspace.

Let \( 0 \neq x \in \mathcal{E}_\lambda \), then \( Tx \in \mathcal{E}_\lambda \) also as \( Tx = \lambda x \) and \( T(\lambda x) = \lambda (\lambda x) \).

Example from last exercise: \( T \{ \text{constant functions} \} = \{ \text{constant functions} \} \).

7.2.6

Suppose \( T_1 \) is an extension of \( T \), i.e., \( D(T) \subset D(T_1) \) and \( T_1 x = T x \) for all \( x \in D(T) \). Let \( \lambda \in \sigma(p(T)) \), then there exists \( 0 \neq x \in D(T) \) such that \( T x = \lambda x \), but then \( T_1 x = T x = \lambda x \) also, so \( \lambda \in \sigma(T_1) \). Let \( \mathcal{E}_\lambda \) be the eigenspace corresponding to the eigenvalue \( \lambda \) of \( T \), and \( \mathcal{E}_{\lambda}' \) similar for \( T_1 \).

Suppose \( x \in \mathcal{E}_\lambda \), i.e., \( T x = \lambda x \), then \( T_1 x = T x = \lambda x \), so \( x \in \mathcal{E}_{\lambda}' \) also.
Let \( \text{PWLC}[a,b] \) be the space of piecewise linear continuous functions on \([a,b]\). Clearly \( f, g \) are both in \( \text{PWLC}[a,b] \), if \( f \) and \( g \) do. For example, not that elements \( f \in \text{PWLC}[a,b] \) are defined by their values on a finite set of points \( \{ a \leq x_0 < x_1 < \cdots < x_n < b \} \). Thus \( f, g \) can be constructed from a finite set of points corresponding to \( f, g \) and the points of intersection.

Moreover, for each \( h \in \text{C}[a,b] \) and \( a \leq x_1 < x_2 \leq b \), there exists a linear function \( f \in \text{PWLC}[a,b] \) s.t. \( f(x_1) = h(x_1), f(x_2) = h(x_2) \),

By applying Lemma 2 of the Stone-Weierstrass Theorem, note, for all \( \varepsilon > 0 \), there exists \( g \in \text{PWLC}[a,b] \) s.t. \( \|f - g\|_{\infty} < \varepsilon \).