7.1.10 Let $\lambda \in \mathbb{C}$ and $\lambda_1, \dots, \lambda_k$ the roots of the polynomial $p - \lambda$,

i.e. $p(x) - \lambda = \alpha(x - \lambda_1) \cdots (x - \lambda_k)$, so that

$p(A) - \lambda$ is invertible if and only if $A - \lambda_i$ is invertible for all i.

That is $\lambda \notin \sigma(p(A)) \iff \lambda_i \notin \sigma(A)$ for all $i \iff \lambda \notin p(\sigma(A))$

(since $p(\lambda_i) = \lambda$ for all i)

7.1.15 $X = \{$ polynomials $x(t) = a_0 + a_1 t + \cdots + a_{n-1} t^{n-1} \}$

$T: X \to X$, $(Tx)(t) = x'(t) = a_1 + 2a_2 t + \cdots + (n-1) a_{n-1} t^{n-2}$

Clearly $(Tx)(t) = x'(t) = \lambda x(t) \iff \lambda = 0$ and $x(t) = a_0$ for some constant.

Hence 0 is an eigenvalue of algebraic multiplicity n, and geometric multiplicity 1 (the eigenspace E_0 corresponding to $\lambda = 0$ is 1-dimensional, consisting of the constant functions)

7.2.3 Suppose λ is an eigenvalue of T and E_λ the corresponding eigenspace

Let $0 \neq x \in E_\lambda$, then $Tx \in E_\lambda$ also as $Tx = \lambda x$ and $T(\lambda x) = \lambda(\lambda x)$.

Example from last exercise: $T\{$constant functions$\} = \{0\} \subset \{$constant functions$\}$

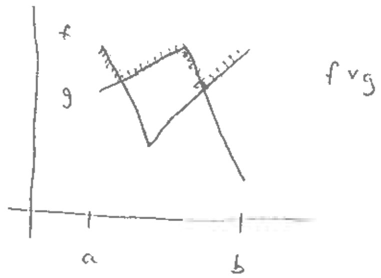
7.2.6 Suppose T_1 is an extension of T, i.e. $\mathcal{D}(T_1) \supset \mathcal{D}(T)$ and $T_1 x = Tx$ for all $x \in \mathcal{D}(T)$. Let $\lambda \in \sigma_p(T)$, then there exists $0 \neq x \in \mathcal{D}(T)$ such that $Tx = \lambda x$, but then $T_1 x = Tx = \lambda x$ also, so $\lambda \in \sigma_p(T_1)$. Let E_λ be the eigenspace corresponding to the eigenvalue λ of T, and E_λ^1 similar for T_1.

Suppose $x \in E_\lambda$, i.e. $Tx = \lambda x$, then $T_1 x = Tx = \lambda x$, so $x \in E_\lambda^1$ also.

Extra

Let PWLC$[a,b]$ be the space of piecewise linear cont. functions on $[a,b]$

Clearly $f \vee g$ and $f \wedge g$ belong to PWLC$[a,b]$, if f and g do. For example, not that elements $f \in$ PWLC$[a,b]$ are defined by their values on a finite set of points $\{a < x_1 < \cdots < x_n < b\}$. Thus $f \vee g$ and $f \wedge g$ can be constructed from a finite set of points corresponding to f, g and the points of intersection.



Moreover for each $h \in C[a,b]$ and $a \leq x_1 < x_2 \leq b$, there exists a linear function $f \in$ PWLC$[a,b]$ s.t. $f(x_1) = h(x_1)$, $f(x_2) = h(x_2)$

By applying Lemma 2 of the Stone-Weierstrass Theorem notes, for all $\varepsilon > 0$, there exists $g \in$ PWLC$[a,b]$ s.t. $\|g - h\|_\infty < \varepsilon$.