1) Let $Y$ be dense in $X$, i.e. $\overline{Y} = X$ and let $A$ be (nonempty) open in $X$.

Clearly $Y \cap A \subset A$ so $\overline{Y \cap A} \subset \overline{A}$, so suppose $x \in A$, but $x \notin \overline{Y \cap A}$.

Then there is open $U$ containing $x$ s.t. $U \cap \overline{Y \cap A} = \phi$, so $U \cap (Y \cap A) = \phi$.

But then $Y$ has empty intersection with an open set $U \cap A$ so $Y$ is not dense

This is a contradiction, hence $A \subset \overline{Y \cap A}$, thus $\overline{A} \subset \overline{Y \cap A}$

2) Let $A \subset X$, $B \subset Y$, then $A \times B \subset \overline{A} \times \overline{B}$ which is closed, so $\overline{A \times B} \subset \overline{A} \times \overline{B}$.

Let $(a,b) \in \overline{A} \times \overline{B}$ and let $U, V$ be open sets around $a, b$, then there exists

$x \in U \cap A$ and $y \in V \cap B$, thus $(U \times V) \cap (A \times B) \neq \phi$.

Since this holds for all open neighborhoods $U, V$ of $a, b$, then $(a,b) \in \overline{A \times B}$.

Let $A \subset X$, $B \subset Y$, then $A^\circ \times B^\circ$ is open an $A^\circ \times B^\circ \subset A \times B$ so $A^\circ \times B^\circ \subset (A \times B)^\circ$

Let $(a,b) \in (A \times B)^\circ$. Since $(a,b)$ is an interior point there exists open neighbourhoods

$U$ and $V$ of $a, b$ s.t. $(a,b) \in U \times V \subset A \times B$ (this is a consequence of

the way product topology is defined - sets of form $U \times V$ is a base for the topology).

Then $U \subset A^\circ$ and $V \subset B^\circ$ so $(a,b) \in A^\circ \times B^\circ$.

3) a1: Suppose $f: X \to Y$ is continuous and $Y$ Hausdorff, $G(f) = \{(x,y) \in X \times Y : f(x) = y\}$

Want to show that $G(f)^c = (X \times Y) \setminus G(f)$ is open

Pick $(x,y) \in G(f)^c$ and let $z = f(x)$, then (since $Y$ is Hausdorff)

there exists open sets $U_y$ and $U_z$ around $y$ and $z$ s.t. $U_y \cap U_z = \emptyset$

Let $V = f^{-1}(U_z)$, then $V$ is open and contains $x$, so $V \times U_y$ is an

open set in $X \times Y$ containing $(x,y)$. Also $V \times U_y \subset G(f)^c$ since

$f(V) \subset U_z$ so $f(V) \cap U_y = \emptyset$.

b1: Suppose $Y$ is compact, ~~Hausdorff~~ and $G(f)$ is closed

Want to show that for each $x \in X$ and open set $V$ around $y = f(x)$

there exists an open set $U$ containing $x$ s.t. $f(U) \subset V$.

Note that the set $B = (X \times V^c) \cap G(f)$ is closed in $X \times Y$ and that

$\{x\} \times Y \cap B = \emptyset$ (since $f(x) \in V$). Want to find an open set $U$ of $x$

s.t. $U \times Y \cap B = \emptyset$, then $f(U) \subset V$.

[Note that $B = \{(x, f(x)) : x \in X, f(x) \notin V\}$ and the projection of $B$

onto $X$, $B_x = \{x \in X : f(x) \notin V\}$ is closed since $Y$ is compact, pick $U = B_x^c$]

For each $z \in Y$, let $W_z = U_z \times V_z$ be an open set around $(x,z)$ in $X \times Y$

s.t. $U_z, V_z$ are open around $x, z$ and $W_z \cap B = \emptyset$.

$\bigcup_{z \in Y} V_z$ is an open cover of $Y$, and by compactness there is a finite subcover

$\bigcup_{i=1}^{k} V_{z_i}$ of $Y$ and if $U = \bigcap_{i=1}^{k} U_{z_i}$, then $U \times Y \cap B = \emptyset$.

b2: Suppose $X, Y$ compact Hausdorff, $G(f)$ closed in $X \times Y$ (don't need $Y$ Hausdorff).

Let $V \subset Y$ be open, then $B$ (as above) is closed, hence compact.

The projection $(x,y) \mapsto x$ is continuous, so $B_x$ is therefore compact,

hence closed since $X$ is Hausdorff. Then $B_x^c = f^{-1}(V)$ is open.

a2: Consider the map $X \times Y \xrightarrow{\varphi} Y \times Y$ given by $(x,y) \overset{\varphi}{\mapsto} (f(x), y)$ and let

$\Delta(Y) = \{(y,y) : y \in Y\}$. Then $\Delta(Y)$ is closed as $Y$ is Hausdorff and $\varphi$ continuous,

so $G(f) = \varphi^{-1}(\Delta(Y))$ is closed.

4) Let $g_i = f - f_i$, so $i \le j \Rightarrow g_i(x) \ge g_j(x)$ and $\lim_i g_i(x) = 0$

For each $x$, $\varepsilon > 0$ there exists $N = N(x)$ s.t. $g_N(x) < \varepsilon$, and since $g_N$ is continuous there exists open $U = U(x)$ s.t. $g_N(y) < \varepsilon$ for all $y \in U$.

Moreover $\bigcup_{x \in X} U(x)$ is an open cover for $X$ so there exists a finite cover $\bigcup_{i=1}^{k} U(x_i)$ of $X$ as $X$ is compact. If $N \ge N(x_1), \dots, N(x_k)$, then $g_N(x) < \varepsilon$ for all $x$ and $n \ge N$ implies $g_n(x) \le g_N(x)$.

Hence $\sup_x |g_i(x)| \to 0$