4.12.5)

T is linear: $T(\alpha x + \beta y)_n = \frac{1}{n}(\alpha x_n + \beta y_n) = \alpha \frac{1}{n} x_n + \beta \frac{1}{n} y_n = (\alpha T x)_n + (\beta T y)_n$

T is bounded: $\|T\| = \sup_{\|x\|=1} \|Tx\| = \sup_{\|x\|=1} \sup_j |\frac{1}{j} x_j| \leq 1$

$T^{-1}x = (x_1, 2x_2, 3x_3, \dots)$, and for each N there exists a sequence x (e.g. $x = \sum_{i=1}^{N+1} e_i$) s.t. $\|x\| = 1$ but $\|T^{-1}x\| > N$.

Hence T^{-1} is not bounded. This does not contradict the open mapping theorem since X is not complete

4.12.6)

Suppose $T^{-1}: \operatorname{ran} T \to X$ is bounded, and let $(y_n) \subset \operatorname{ran} T$ be a sequence such that $y_n \to y \in Y$. Then $(x_n) = (T^{-1} y_n)$ is a Cauchy sequence in X, i.e. $\|x_n - x_m\| = \|T^{-1}y_n - T^{-1}y_m\|$ $\leq \|T^{-1}\| \|y_n - y_m\| \to 0$, so $\lim_n x_n = x \in X$ since X is complete and $Tx = \lim Tx_n = \lim y_n = y$, so $y \in \operatorname{ran} T$

Suppose $\operatorname{ran} T$ is closed, which means that $\operatorname{ran} T$ is complete (since Y is complete). Hence $T: X \to \operatorname{ran} T$ is a bounded linear bijection between Banach spaces, and the open mapping theorem says T^{-1} is bounded

4.12.8)

Let $I: X_1 \to X_2$ be the identity map. If $\|x_n\|_1 \to 0 \Rightarrow \|x_n\|_2 \to 0$, then I is bounded (if I is not bounded there exists a sequence (x_n) s.t. $\|x_n\|_1 = 1$ but $\|Ix_n\|_2 > n$, that is $(\frac{1}{n} x_n) \xrightarrow{1} 0$, but $\|\frac{1}{n} x_n\|_2 > 1$ for all n). Then $\|x\|_2 = \|Ix\|_2 \leq b\|x\|_1$ for some b.

Hence $I^{-1}: X_2 \to X_1$ is also bounded (by open mapping theorem), so there exists c s.t. $\|x\|_1 = \|Ix\|_1 \leq c\|x\|_2$. Set $c = \frac{1}{a}$ to get

$a\|x\|_1 \leq \|x\|_2 \leq b\|x\|_1$

So if $\|x_n\|_2 \to 0$, then $\|x_n\|_1 \leq c\|x_n\|_2 \to 0$ also. Then $x_n \xrightarrow{1} x$ $\iff x_n \xrightarrow{2} x$ as $\|x_n - x\|_1 \to 0 \iff \|x_n - x\|_2 \to 0$

4.12.9) Let $I: Y_2 \longrightarrow X_1$ be the identity map. If there is a constant c s.t. $\|Ix\|_1 = \|x\|_1 \leq c\|x\|_2$, then I is bounded and $\|I\| \leq c$. Then $I^{-1}: Y_1 \longrightarrow X_2$ is also bounded, i.e. there exists a k such that $\|x\|_2 = \|I^{-1}x\|_2 \leq k \cdot \|x\|_1$.

by the Open Mapping Theorem

(same exercise as 4.12.8 ...)

4.12.10) Suppose $\mathcal{J}_1 \supset \mathcal{J}_2$, i.e. $U \in \mathcal{J}_2 \Rightarrow U \in \mathcal{J}_1$ for all open sets U.

Then the map $I: X_1 \longrightarrow X_2$ is continuous, i.e. bounded. The Open Mapping theorem now states that $I^{-1}: X_2 \longrightarrow X_1$ is bounded, which means that $U \in \mathcal{J}_1 \Rightarrow U \in \mathcal{J}_2$, that is $\mathcal{J}_1 = \mathcal{J}_2$

4.13.11) Suppose $T: X \longrightarrow Y$ is a closed linear operator. Let (x_n) be a sequence in $\ker T$ s.t. $x_n \longrightarrow x \in X$, then (by Thm 4.13-3, since $Tx_n \longrightarrow 0 = y$) $Tx = y = \lim Tx_n = 0$, so $x \in \ker T$.

4.13.14) Let $u_i \in C'[0,1]$, $x_N = \sum_{i=1}^{N} u_i \xrightarrow{N \to \infty} x$ uniformly

Also assume $y_N = \sum_{i=1}^{N} u_i'$ converges uniformly on $[0,1]$ to y.

The map $T: C'[0,1] \longrightarrow C[0,1]$, $x \longmapsto x'$ is closed (see example 4.13-4) and then $x \in C'[0,1]$ and $x' = Tx = y$.