4.5.2)

$(0^\times g)(x) = g(0x) = g(0) = 0$, so $0^\times$ is the zero operator

$(I^\times g)(x) = g(Ix) = g(x)$, so $I^\times$ is the identity operator

4.5.9)

$T: X \to Y$ bounded linear operator and $M = \overline{\operatorname{ran} T}$, show $M^a = \ker T^\times$

Assume $g \in M^a = \{f \in Y' : f(M) = 0\}$, then $g(M) = 0$, in particular

$g(\operatorname{ran} T) = 0$, i.e. $g(Tx) = 0$ for all $x \in X$, hence $T^\times g = 0$. Thus $M^a \subset \ker T^\times$

Assume $g \in \ker T^\times$, i.e. $g(Tx) = 0$ for all $x \in X$, hence $g(\operatorname{ran} T) = 0$

The continuity of g then implies that $g(M) = 0$.

Indeed, let $y \in \overline{\operatorname{ran} T}$, then there exists $(y_i) \subset \operatorname{ran} T$ s.t. $y_i \to y$,

hence $g(y) = g(\lim y_i) = \lim g(y_i) = 0$.

4.5.10)

Show $\operatorname{ran} T \subset {}^a\ker T^\times = \{y \in Y : g(y) = 0 \text{ for all } g \in \ker T^\times\}$

Assume $y \in \operatorname{ran} T$, i.e. $y = Tx$ for some $x \in X$ and suppose $g \in \ker T^\times$.

Then $g(y) = g(Tx) = (T^\times g)(x) = 0$, so $g \in {}^a\ker T^\times$.

(Note that $M = {}^a\ker T^\times$, in general ${}^a(N^a) = \overline{N}$)

4.6.4)

Suppose X is reflexive, i.e. $X'' = X$, then $(X')'' = (X'')' = X'$,

so that X' is reflexive. (The map $C: X' \to X'''$ given by

$Cf(\mathcal{X}_x) = f(x)$, $x \longleftrightarrow \mathcal{X}_x$ identification $X \longleftrightarrow X''$)

Suppose X' is reflexive, then X'' is reflexive by the argument above.

Since X is a closed subspace of X'', then X is also reflexive.

4.6.7)

Let Y be a closed subspace of a normed space X, and suppose $Y \neq X$.

Pick $x_0 \in X \setminus Y$, then (by Lemma 4.6-7) there exists $f \in X'$,

s.t. $f(Y) = 0$ and $f(x_0) \neq 0$.

Hence $Y = X$

Extra 1)

$$T: \ell^1 \to c_0 \quad , \quad T^\times : \ell^1 \to \ell^\infty$$

$$(Tx)_n = \sum_{m=n}^{\infty} x_m = \sum_{m=0}^{\infty} x_{m+n}$$

T linear: $T(\alpha x + \beta y)_n = \sum_{m=n}^{\infty} \alpha x_m + \beta y_m = \alpha \sum_{m=n}^{\infty} x_m + \beta \sum_{m=n}^{\infty} y_m$

$$= \alpha (Tx)_n + \beta (Ty)_n$$

T bounded: $\|Tx\|_\infty = \sup_n |(Tx)_n| = \sup_n \left| \sum_{m=n}^{\infty} x_m \right| \leq \sup_n \sum_{m=n}^{\infty} |x_m|$

$$\leq \sum_{m=1}^{\infty} |x_m| = \|x\|_1$$

Let $f \in (c_0)' = \ell^1$ represented by the sequence $(y_i) \subset \ell^1$, $y_i = f(e_i)$

Then $(T^\times f)(x) = f(Tx) = \sum_{n=1}^{\infty} (Tx)_n y_n = \sum_{n=1}^{\infty} \left(\sum_{m=n}^{\infty} x_m \right) y_n$

$$= \sum_{n=1}^{\infty} x_n \left(\sum_{m=1}^{n} y_m \right)$$

Thus $(T^\times y)_n = \sum_{m=1}^{n} y_m$

Extra 2)

Check out example 2.10.7 + exercise 2.10.8 (week 2)

$f \in (\ell^p)'$ is identified with $y = (y_i) = (f(e_i)) \subset \ell^q$, $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p, q < \infty$

Must show (i) $y \in \ell^q$, (ii) $\|y\|_q \leq \|f\|_{(\ell^p)'}$ (iii) $\|f\|_{(\ell^p)'} \leq \|y\|_q$

Let $x_k = \sum_{j=1}^{k} \frac{|y_j|^q}{y_j}$, then $\sum_{j=1}^{k} |y_j|^q = f(x_k) \leq \|f\| \left(\sum |y_j|^k \right)^{1/p}$

↳ check

hence $\|y\|_q \leq \|f\|$, showing $y \in \ell^q$ and (ii)

Use Hölder's inequality to show (iii)

Thus $(\ell^p)'' = (\ell^q)' = \ell^p$