4.5.2

\[(\mathcal{L}^* g)(x) = g(0x) = g(0) = 0, \text{ so } 0^* \text{ is the zero operator.}\]

\[i^* g)(x) = g(Ix) = g(x). \text{ so } I^* \text{ is the identity operator.}\]

4.5.9

Assume \(g \in M^a = \text{ker } T^a\), then \(g(M) = 0\), in particular \(g(\text{ran } T) = 0\), i.e. \(g(Tx) = 0\) for all \(x \in X\), hence \(T^a g = 0\). Thus \(M^a \subseteq \text{ker } T^a\).

Assume \(g \in \text{ker } T^a\), i.e. \(g(Tx) = 0\) for all \(x \in X\), hence \(g(\text{ran } T) = 0\). The continuity of \(g\) then implies that \(g(M) = 0\).

Indeed, let \(y \in \text{ran } T\), then there exists \((y_i) \in \text{ran } T\) s.t. \(y_i \to y\), hence \(g(y) = g(\lim y_i) = \lim g(y_i) = 0\).

4.5.10

Show \(\text{ran } T \subseteq \text{ker } T^a \subseteq \{ y \in Y : g(y) = 0 \text{ for all } g \in \text{ker } T^a \}\).

Assume \(y \in \text{ran } T\), i.e. \(y = Tx\) for some \(x \in X\) and suppose \(g \in \text{ker } T^a\).

Then \(g(y) = g(Tx) = (T^a g)(x) = 0\), so \(g \in \text{ker } T^a\).

(Note that \(M = \text{ker } T^a\), in general \(\dim(M) \leq N\).

4.6.1

Suppose \(X\) is reflexive, i.e. \(X'' = X\), then \((X')' = (X'')' = X\)', so that \(X'\) is reflexive. (The map \(C: X' \to X''\) given by \(Cf(x) = f(x), \quad x \in X\), identify \(X \to X''\))

Suppose \(X'\) is reflexive, then \(X''\) is reflexive by the argument above.

Since \(X\) is a closed subspace of \(X''\), then \(X\) is also reflexive.

4.6.7

Let \(Y\) be a closed subspace of a normed space \(X\), and suppose \(Y \neq X\).

Pick \(y_0 \in X \setminus Y\), then (by Lemma 4.6.7) there exists \(f \in X'\), s.t. \(f(y) = 0\) and \(f(x_0) \neq 0\).

Hence \(Y = X\).
Ex. 1)

\[ T : l^1 \rightarrow c_0, \quad T^* : l^\infty \rightarrow l^1 \]

\[ (T^*)_n : \sum_{m=0}^{\infty} x_m = \sum_{m=0}^{n} x_m \]

\[ T \text{ linear : } T(\alpha x + \beta y)_n = \sum_{m=n}^{\infty} \alpha x_m + \beta y_m = \alpha \sum_{m=n}^{\infty} x_m + \beta \sum_{m=n}^{\infty} y_m \]

\[ = \alpha (T^*)_n + \beta (T_y)_n \]

\[ T \text{ bounded : } \|T^* x\|_\infty = \sup_n |(T^* x)_n| = \sup_n |\sum_{m=n}^{\infty} x_m| \leq \sup_n \sum_{m=n}^{\infty} |x_m| \]

\[ \leq \sum_{m=n}^{\infty} |y_m| = \|x\|_1 \]

Let \( f \in (c_0)' = l^1 \) represented by the sequence \( (\gamma_i)_i \in l^1 \), \( \gamma_i = f(\xi_i) \)

Then \( (T^* f)(x) = f(T^* x) = \sum_{n=1}^{\infty} (T^* x)_n \gamma_n = \sum_{n=1}^{\infty} \left( \sum_{m=n}^{\infty} x_m \right) \gamma_n \)

\[ = \sum_{n=1}^{\infty} x_n \left( \sum_{m=n}^{\infty} \gamma_m \right) \]

Thus \( (T^* f)_n = \sum_{m=n}^{\infty} \gamma_m \)

Ex. 2)

Check with example 2.10.7 + exercise 2.10.8 (week 2)

\( f \in (l^p)' \) is identified with \( g : y_i = f(y_i) \in l^q, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad 1 \leq p, q \leq \infty \)

Must show:

(i) \( y \in l^q \),

(ii) \( \|y\|_q \leq \|f\|_{(l^p)'} \),

(iii) \( \|f\|_{(l^p)'} \leq \|y\|_q \)

Let \( x_k = \sum_{j=1}^{k} |y_j|^{1/p} \), then \( \sum_{j=1}^{k} |y_j|^{1/p} = f(y_k) \leq \|f\|_{(l^p)'} \left( \sum_{j=1}^{k} |y_j|^p \right)^{1/p} \)

hence \( \|y\|_q \leq \|f\|_{(l^p)'} \), showing \( y \in l^q \) and (i).

Use Hölder's inequality to show (ii).

Thus \( (l^p)'' = (l^q)' = l^p \).