3.8.5 Consider the map $\ell^2 \longmapsto (\ell^2)^*$ given by $z \longmapsto f_z$, $f_z(x) = \langle x, z \rangle$

Linear: $f_{\alpha z + \beta v}(x) = \langle x, \alpha z + \beta v \rangle \underset{\text{since real } \ell^2}{=} \langle \alpha z + \beta v, x \rangle = \alpha \langle z, x \rangle + \beta \langle v, x \rangle$

$= \alpha \langle x, z \rangle + \beta \langle x, v \rangle = \alpha f_z(x) + \beta f_v(x)$

Isometry: $\| f_z \| = \sup_{\|x\|=1} |f_z(x)| = \sup_{\|x\|=1} |\langle x, z \rangle| \overset{\text{Cauchy-Schwarz}}{\leq} \sup_{\|x\|=1} \|x\| \cdot \|z\| = \|z\|$

$\| f_z \| = \sup_{\|x\|=1} |f_z(x)| = \sup_{\|x\|=1} |\langle x, z \rangle| \geq \left| \left\langle \frac{z}{\|z\|}, z \right\rangle \right| = \frac{1}{\|z\|} \cdot \|z\|^2 = \|z\|$

Surjection: Follows by Riesz's theorem, for every $f_z \in (\ell^2)^*$, there exists unique $z \in \ell^2$ s.t. $f_z(x) = \langle x, z \rangle$ for all $x \in \ell^2$

(Injection follows from isometry)

3.8.6 Consider the map $H \longmapsto H^*$ given by $z \longmapsto f_z$, $f_z(x) = \langle x, z \rangle$

Linearity: $f_{\alpha z + \beta v}(x) = \langle x, \alpha z + \beta v \rangle = \overline{\langle \alpha z + \beta v, x \rangle} = \overline{\alpha \langle z, x \rangle + \beta \langle v, x \rangle}$

$= \bar{\alpha} \langle x, z \rangle + \bar{\beta} \langle x, v \rangle = \bar{\alpha} f_z(x) + \bar{\beta} f_v(x)$

Isometry: $\left.\begin{array}{l} \| f_z \| \leq \|z\| \text{ by Cauchy-Schwarz} \\ \| f_z \| \geq \|z\| \end{array}\right\}$ as above

Surjection follows from Riesz' theorem and injection from the fact that it is an isometry.

3.8.8 Define $T: H \longrightarrow H^*$ and $S: H^* \longrightarrow H^{**}$ by $z \longmapsto f_z$

Then ST is linear as $ST(\alpha z + \beta v) = S(\bar{\alpha} Tz + \bar{\beta} Tv) = \alpha STz + \beta STv$

Since S and T are bijective isometries, so is ST $(\|STz\| = \|Tz\| = \|z\|)$

Note that $f_{f_z}(f_v) = \langle f_v, f_z \rangle = \langle z, v \rangle = f_v(z)$, so every $\chi \in H^{**}$ is given by evaluation, i.e. $\chi(f) = f(z)$ for some $z \in H$, and ST is the map $z \longmapsto \chi_z$

Moreover $\langle \chi_z, \chi_v \rangle = \langle f_v, f_z \rangle = \langle z, v \rangle$.

Remark H and H^* are anti-isomorphic as Hilbert spaces, but isomorphic as Banach spaces

3.9.3 Since $\|S\| = \|S^*\|$ for all $S \in B(H)$, for all $\varepsilon > 0$ there exist N s.t.

$\|T^* - T_n^*\| = \|(T - T_n)^*\| = \|T - T_n\| < \varepsilon$ for all $n > N$,

i.e. $T_n^* \longrightarrow T$.

3.9.10 Define $T(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$

Clearly $\operatorname{ran}(T) = \{x \in H \mid x_1 = 0\}$ and $\ker(T) = \{0\}$

Moreover $\|Tx\|^2 = \sum_i |\langle Tx, e_i\rangle|^2 = \sum_i |x_i|^2 = \sum_i |\langle x, e_i\rangle|^2 = \|x\|^2$,

by Parseval, so $\|T\| = 1$

Furthermore, $\langle Tx, e_n\rangle = \langle x, T^* e_n\rangle \Rightarrow \left\langle \sum_i \langle Tx, e_i\rangle e_i, e_n\right\rangle = \left\langle \sum_i \langle x, e_i\rangle e_i, T^* e_n\right.$

so that $x_{n-1} = \sum_i x_i (T^* e_n)_i$ for $n \geq 2$, hence $T^* e_n = e_{n-1}$.

Also $0 = \langle Tx, e_1\rangle = \langle x, T^* e_1\rangle$, i.e. $T^* e_1 = 0$

We then have that $T^*(x_1, x_2, \dots) = (x_2, x_3, \dots)$.

Note that $T^*T = I$, but $TT^* \neq I$.

Extra Let $w \in C^1[a,b]$, then

$$\operatorname{Var}(w) = \sup_P \sum_i |w(t_j) - w(t_{j-1})| = \sup_P \sum_i |w'(c_j)(t_j - t_{j-1})|$$

$$\leq \sup_{x \in [a,b]} |w'(x)| \cdot \sup_P \sum_i |t_j - t_{j-1}| \leq \|w\|_1 (b-a)$$

Here we apply the mean value theorem; there exists $c_j \in (t_{j-1}, t_j)$ s.t.

$$w'(c_j) = \frac{w(t_j) - w(t_{j-1})}{t_j - t_{j-1}}$$

Definition of integrals: For all $\varepsilon > 0$, there exists $\delta > 0$ s.t. if P_n is a partition with $\max |t_j - t_{j-1}| < \delta$ then

$$\left| \sum_i f(x_j)(w(x_j) - w(x_{j-1})) - \int f\,dw \right| < \varepsilon \quad \text{and} \quad \left| \sum_i f(x_j) w'(x_j)(x_j - x_{j-1}) - \int f w'\,dx \right| < \varepsilon$$

Idea: Estimate $\left| \sum_i f(x_j)(w(x_j) - w(x_{j-1})) - \sum_i f(x_j) w'(x_j)(x_j - x_{j-1}) \right|$

By MVT there exist $c_j \in (x_{j-1}, x_j)$ s.t. $w(x_j) - w(x_{j-1}) = w'(c_j)(x_j - x_{j-1})$

and by continuity of w' the $w'(c_j)$'s will be 'close' to the $w'(x_j)$'s

Indeed, for each $\varepsilon > 0$ there exists $\delta_j > 0$ s.t.

$|w'(x_j) - w'(c_j)| < \varepsilon$ whenever $|x_j - c_j| < \delta_j$

Thus for each $\varepsilon > 0$, we can pick a $\delta > 0$ s.t. (pick δ as min. of δ_j and δ from def

$$\max |x_j - x_{j-1}| < \delta \Rightarrow \left| \int f\,dw - \int f w'\,dx \right| < \left| \sum_i f(x_j)(w'(c_j) - w'(x_j))(x_j - x_{j-1}) \right| + 2\varepsilon$$

$$< \|f\| \cdot \varepsilon \cdot \sum_i (x_j - x_{j-1}) + 2\varepsilon = ((b-a)\|f\| + 2)\varepsilon$$

Hence $\int f\,dw = \int f w'\,dx$