Consider the map \( f_2 : (l^2)^4 \to l^2 \) given by \( \mathbf{x} \mapsto f_2(\mathbf{x}) = \langle x, z \rangle \)

**Linearity:**
\[
\langle f_2(x), \beta \rangle = \langle x, \beta \rangle = \langle x, z \rangle + \langle \alpha, z \rangle = \langle f_2(x), \alpha \rangle + \langle f_2(x), \beta \rangle
\]

**Isometry:**
\[
\| f_2 \| = \sup_{\| x \|_2 = 1} \langle f_2(x), 1 \rangle \leq \sup_{\| x \|_2 = 1} \| x \|_2 = 1 \| x \|_2 = 1 \| x \|_2 = 1
\]

**Injective:**
By Riesz's theorem, for every \( f_2 \in (l^2)^4 \), there exists a unique \( z \in l^2 \) such that \( f_2(x) = \langle x, z \rangle \) for all \( x \in l^2 \). (Injection follows from isometry)

3.8.6 Consider the map \( H \mapsto H^* \) given by \( \mathbf{x} \mapsto f_2(\mathbf{x}) = \langle x, z \rangle \)

**Linearity:**
\[
\langle f_2(x), \beta \rangle = \langle x, \beta \rangle = \langle x, z \rangle + \langle \alpha, z \rangle = \langle f_2(x), \alpha \rangle + \langle f_2(x), \beta \rangle
\]

**Isometry:**
\[
\| f_2 \| = \sup_{\| x \|_2 = 1} \langle f_2(x), 1 \rangle \leq \sup_{\| x \|_2 = 1} \| x \|_2 = 1 \| x \|_2 = 1
\]

**Injective:**
By Riesz's theorem and isometry from the fact that \( H^* \) is an isometry.

3.8.8 Define \( T : H \to H^* \) and \( S : H^* \to H^* \) by \( \mathbf{x} \mapsto f_2(\mathbf{x}) \)

Then \( ST \) is linear as \( ST(\mathbf{x}, \beta) = S(\mathbf{x}, T \beta) = \alpha \mathbf{f}_2 + \beta S \mathbf{f}_2 \)

Since \( S \) and \( T \) are bijective isometries, so is \( ST \) \( (\| ST \| = \| ST \| = 1) \)

Note that \( \mathbf{f}_2(\mathbf{f}_2) = \langle \mathbf{f}_2, \mathbf{f}_2 \rangle = \langle \mathbf{2}, \mathbf{2} \rangle = \mathbf{2} \), so every \( \mathbf{x} \in H^* \)

is given by evaluation, i.e., \( \mathbf{f}(\mathbf{x}) = \mathbf{f}(z) \) for some \( z \in H \), and \( ST \)

is the map \( \mathbf{x} \mapsto \mathbf{x} \).

Moreover \( \langle \mathbf{x}, \mathbf{f}_2 \rangle = \langle \mathbf{f}_2, \mathbf{f}_2 \rangle \).

**Remark**
\( H \) and \( H^* \) are anti-symmetric as Hilbert spaces, but isomorphic as Banach spaces.

3.9.3 Since \( \| ST \| = \| ST \| \) for all \( \mathbf{f} \in \mathcal{B}(H) \), for all \( \mathbf{x} \in H \) and \( N \in \mathbb{N} \),

\[
\| T^{-N} - T_+^{-N} \| = \| (T - T_+) \| \leq E \quad \text{for all } N \in \mathbb{N},
\]

i.e., \( T^{-N} \to T \).
3.9.10

Define \( T(x_1, x_2, \ldots) = (c, x_1, x_2, \ldots) \)

Clearly \( \text{ran}(T) = \{x \in H \mid x_i = 0 \} \) and \( \ker(T) = \{0\} \)

Moreover, \( \|T x \|_1^2 = \sum_i |<Tx, e_i>|^2 = \sum_i |x_i|^2 = \sum_i |e_i|^2 = 1 = \|x\|_1^2 \)

by Parseval, so \( \|T\| = 1 \)

Furthermore, \( <Tx, e_n> = <x, T^* e_n> \Rightarrow \sum_i <Tx, e_i> e_i \equiv \sum_i <x, e_i> e_i, T^* e_n \)

so that \( x_n = \sum_i X_i (T^* e_n) \). For \( n \geq 2 \), hence \( T^* e_n = e_{n-1} \)

Also \( 0 = <Tx, e_1> = <x, T^* e_1> = r + T^* e_0 = 0 \)

We then have that \( T^* (x_1, x_2, \ldots) = (x_1, x_2, \ldots) \).

Note that \( T^* T = I \), but \( TT^* \neq I \).

Extra

Let \( w \in C[a, b] \), then

\[ \text{Var}(w) = \frac{\sum_i |w(t_i) - w(t_{i-1})|}{\sum_i |w(t_i) - w(t_{i-1})|} \geq \frac{\sum_i |w(t_i) - w(t_{i-1})|}{\sum_i |t_i - t_{i-1}|} \in \mathbb{N} \]

Here we apply the mean value theorem: there exists \( c_i \in (t_{i-1}, t_i) \) such that \( w'(c_i) = \frac{w(t_i) - w(t_{i-1})}{t_i - t_{i-1}} \).

Definition of integrals: For all \( x > 0 \), there exists \( S > 0 \) such that \( \rho_n \)

is a polynomial with \( \max |t_i - t_{i-1}| < 6 \) then

\[ |\sum f(x_i)(w(x_i) - w(x_{i-1})) - \int f(w) dx| < \varepsilon \quad \text{and} \quad |\sum f(x_i)w'(x_i)(x_i - x_{i-1}) - \int f(w') dx| < \varepsilon \]

Idea: Estimate \( |\sum f(x_i)(w(x_i) - w(x_{i-1})) - \sum f(x_i)w'(x_i)(x_i - x_{i-1})| \)

By NVT there exists \( c_i \in (x_{i-1}, x_i) \) s.t. \( w(x_i) - w(x_{i-1}) = w'(c_i)(x_i - x_{i-1}) \)

and by continuity of \( w' \) the \( w'(c_i) \)'s will be 'close' to the \( w'(x_i) \)'s.

Indeed, for each \( \varepsilon > 0 \) there exists \( \delta > 0 \) s.t.

\[ |w'(x) - w'(c_i)| < \varepsilon \text{ whenever \( |x_i - c_i| < \delta \)} \]

Thus for each \( \varepsilon > 0 \), we can pick a \( \delta > 0 \) s.t. (pick \( \delta = \min \{\delta_1, \delta_2\} \))

\[ |\sum f(x_i)(w(x_i) - w(x_{i-1})) - \sum f(x_i)w'(x_i)(x_i - x_{i-1})| < \varepsilon \]

\[ \leq \|f\|_1 \cdot \varepsilon \sum |x_i - x_{i-1}| + 2\varepsilon = ((b-a)\|f\|_1 + 2)\varepsilon \]

Hence \( \int f dw = \int f w'dx \)