

4.1.2: Reflexivity: If $x \in X$ then $x(t) \leq x(t)$ for all $t \in [0, 1]$ so $x \leq x$.

Antisymmetry: If $x, y \in X$, $x \leq y$ and $y \leq x$ then $x(t) \leq y(t)$ and $y(t) \leq x(t)$, and thus $x(t) = y(t)$, for any $t \in [0, 1]$ from which it follows that $x = y$.

Transitivity: If $x, y, z \in X$, $x \leq y$ and $y \leq z$ then $x(t) \leq y(t)$ and $y(t) \leq z(t)$, and thus $x(t) \leq z(t)$, for any $t \in [0, 1]$ from which it follows that $x \leq z$. The order is not total; if for example

$$x(t) = \begin{cases} 0 & \text{if } t \leq \frac{1}{2} \\ 1 & \text{if } t > \frac{1}{2} \end{cases}$$

and

$$y(t) = \begin{cases} 1 & \text{if } t \leq \frac{1}{2} \\ 0 & \text{if } t > \frac{1}{2} \end{cases}$$

then $x \not\leq y$ and $y \not\leq x$. X does not have maximal elements because if $x \in X$, then $x \leq \tilde{x}$ and $x \neq \tilde{x}$ where \tilde{x} is defined by

$$\tilde{x}(t) = x(t) + 1$$

4.2.3

$$\overline{\lim}_{n \rightarrow \infty} \xi_n = \limsup_{n \rightarrow \infty} \xi_n = \inf_{n \geq 1} \sup_{m \geq n} \xi_m$$

Let $(\xi_n)_{n \in \mathbb{N}} \in \ell^\infty$ and $\alpha \in [0, \infty[$. Then

$$\begin{aligned} \overline{\lim}_{n \rightarrow \infty} \alpha \xi_n &= \inf_{n \geq 1} \sup_{m \geq n} \alpha \xi_m = \inf_{n \geq 1} \alpha \sup_{m \geq n} \xi_m \\ &= \alpha \inf_{n \geq 1} \sup_{m \geq n} \xi_m = \alpha \overline{\lim}_{n \rightarrow \infty} \xi_n \end{aligned}$$

Let $(\xi_n)_{n \in \mathbb{N}}, (\eta_n)_{n \in \mathbb{N}} \in \ell^\infty$, and let $\varepsilon > 0$.

Choose $n_0 \in \mathbb{N}$ such that

$$\sup_{m \geq n_0} \xi_m + \sup_{m \geq n_0} \eta_m < \overline{\lim}_{n \rightarrow \infty} \xi_n + \overline{\lim}_{n \rightarrow \infty} \eta_n + \varepsilon$$

Then

$$\begin{aligned} \overline{\lim}_{n \rightarrow \infty} (\xi_n + \eta_n) &\leq \sup_{m \geq n_0} (\xi_m + \eta_m) \leq \sup_{m \geq n_0} \xi_m + \sup_{m \geq n_0} \eta_m \\ &< \overline{\lim}_{n \rightarrow \infty} \xi_n + \overline{\lim}_{n \rightarrow \infty} \eta_n + \varepsilon \end{aligned}$$

and it follows that $\overline{\lim}_{n \rightarrow \infty} (\xi_n + \eta_n) \leq \overline{\lim}_{n \rightarrow \infty} \xi_n + \overline{\lim}_{n \rightarrow \infty} \eta_n$.

) 4.2.5

If $x, y \in M$ and $t \in [0, 1]$, then

$$\begin{aligned} P(tx + (1-t)y) &\leq P(tx) + P((1-t)y) \\ &= tP(x) + (1-t)P(y) \\ &\leq t\gamma + (1-t)\gamma = \gamma \end{aligned}$$

So $tx + (1-t)y \in M$.

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4.2.6

Given $x \in X$ and $\varepsilon > 0$, Choose $\delta > 0$ such that

$\|y\| < \delta \Rightarrow |P(y)| < \varepsilon$. If $\|x - y\| < \delta$ then

$$P(y) = P(x + (y - x)) \leq P(x) + P(y - x) \leq P(x) + |P(y - x)| < P(x) + \varepsilon \text{ and}$$

$$P(x) = P(y + (x - y)) \leq P(y) + P(x - y) \leq P(y) + |P(x - y)| < P(y) + \varepsilon \text{ from which it follows that}$$

$|P(x) - P(y)| < \varepsilon$. This shows that P is continuous at x for any $x \in X$.

4.2.10: Let $Z = \{0\}$ and $f: Z \rightarrow \mathbb{R}$ the linear functional defined by $f(0) = 0$. Notice that $P(0) = P(0 \cdot 0) = 0 \cdot P(0) = 0$ so $f(x) \leq P(x)$ for all $x \in Z$. It follows from Thm 4.2.1 that there exists a linear functional $\tilde{f}: X \rightarrow \mathbb{R}$ such that $\tilde{f}(x) \leq P(x)$ for $x \in X$. We then have that $-\tilde{f}(x) = \tilde{f}(-x) \leq P(-x)$ for $x \in X$ so $-P(-x) \leq \tilde{f}(x) \leq P(x)$ for all $x \in X$.

2.8.12: Let $f \neq 0$ be a linear functional. We claim that f is surjective. To see that choose $x \in X$ such that $f(x) \neq 0$.

We then have for any $\alpha \in K$ that $f\left(\frac{\alpha}{f(x)}x\right) = \frac{\alpha}{f(x)}f(x) = \alpha$. This shows that f

is surjective. It follows that $X/\mathcal{N}(f) \cong K$ and hence that $\text{codim } \mathcal{N}(f) = \dim X/\mathcal{N}(f) = 1$.

Since f is surjective, it follows that there is an $x \in X$ such that $f(x) = 1$. We claim that $H_1 = x + \mathcal{N}(f)$. To see that

$H_1 \subseteq X + \mathcal{N}(f)$ let $y \in H_1$. Then $f(y-x) = f(y) - f(x) = 1 - 1 = 0$ so $y-x \in \mathcal{N}(f)$ and

$y = x + (y-x) \in x + \mathcal{N}(f)$. To see that

$x + \mathcal{N}(f) \subseteq H_1$ let $y \in \mathcal{N}(f)$. Then

$f(x+y) = f(x) + f(y) = 1 + 0 = 1$ so $x+y \in H_1$.

Thus $H_1 = x + \mathcal{N}(f)$ is a hyperplane parallel to $\mathcal{N}(f)$.

4.3.14: Define $f: Kx_0 \rightarrow K$ by $f(\alpha x_0) = \frac{\alpha}{r} \|x_0\|$. Then f is a linear functional and $|f(z)| \leq \frac{1}{r} \|z\|$ for any $z \in Kx_0$, so f is also bounded and $\|f\| \leq \frac{1}{r}$. Thus it follows from Thm 4.3.2 that there exists an $\tilde{f} \in X'$ such that $\|\tilde{f}\| = \|f\| < \frac{1}{r}$ and $\tilde{f}(x_0) = f(x_0) = \frac{1}{r} \|x_0\| = 1$. Hence x_0 belongs to the hyperplane $H_0 = \{x \in X \mid \tilde{f}(x) = 1\}$. If $y \in \tilde{B}(0, r)$ then $\operatorname{Re} \tilde{f}(y) \leq |\tilde{f}(y)| \leq \frac{1}{r} \|y\| \leq 1$, so $\tilde{B}(0, r) \subseteq \{x \in X \mid \operatorname{Re} \tilde{f}(x) \leq 1\}$.