2.7.2 Let $T: X \to Y$ be bounded and $U \subset X$ a bounded set,

i.e. there exist C_T, C_U s.t. $\|Tx\| \le C_T \|x\|$ and $\sup_{x \in U} \|x\| \le C_U$

If $x \in U$, then $Tx \in T(U)$ and $\|Tx\| \le C_T C_U$,

hence $\sup_{y \in T(U)} \|y\| = \sup_{x \in U} \|Tx\| \le C_T \cdot C_U$

Suppose T maps bounded sets to bounded sets, and let $S(X) = \{x \in X : \|x\| = 1\}$

Since $S(X)$ is bounded, there exists c s.t. $\sup_{y \in T(S(X))} \|y\| = \sup_{\|x\|=1} \|Tx\| \le c$

If x (nonzero) is arbitrary, then $\frac{x}{\|x\|}$ has norm 1, so that $\|T \frac{x}{\|x\|}\| \le c$,

hence $\|Tx\| \le c \|x\|$.

2.8.9. Let $f \neq 0$ be a linear functional on X and $x_0 \in X \setminus \ker(f)$ ($\mathcal{N}(f)$)

Let $x \in X$ and put $\alpha = \frac{f(x)}{f(x_0)}$, then $f(x - \alpha x_0) = 0$,

so that $x - \alpha x_0 = y$ for some $y \in \ker(f)$.

Suppose $y_1 + \alpha_1 x_0 = x = y_2 + \alpha_2 x_0$, then $\alpha_1 f(x_0) = f(x) = \alpha_2 f(x_0)$,

so $\alpha_1 = \alpha_2$, thus $y_1 = y_2$.

2.10.8 Let $c_0 \subset \ell^\infty$ be the space of sequences converging to 0, equipped with $\|\cdot\|_\infty$

Let $f \in c_0^*$, and $x = (x_i) = \sum_i x_i e_i$, and $y_i = f(e_i)$ so that

$f(x) = \sum_i x_i y_i$.

Let now $x_k = \sum_{j=1}^{k} \frac{|y_j|}{y_j}$, which means $x_k \in c_0$ and $\|x_k\| = 1$ for each k.

Then $f(x_k) = \sum_{i=1}^{k} |y_i| \le \|f\|_{c_0^*}$ for all k, thus $y \in \ell^1$ and $\|y\|_1 \le \|f\|_{c_0^*}$

Moreover $|f(x)| \le \sum |x_i y_i| \le \|x\|_\infty \cdot \sum |y_i| = \|x\|_\infty \cdot \|y\|_1$,

i.e. $\|f\|_{c_0^*} \le \|y\|_1$. So $\|f\|_{c_0^*} = \|y\|_1$ and $f \longrightarrow (y_i)$ therefore

gives an identification between c_0^* and ℓ^1

2.10.13 Let $\emptyset \neq M \subset X$, and $M^a = \{ f \in X^* \mid f(M) = 0 \}$

Linear subspace: (i) $0 \in M^a$ obviously

(ii) $f, g \in M^a$, $\alpha, \beta \in \mathbb{C}$, $x \in M$, then

$(\alpha f + \beta g)(x) = \alpha f(x) + \beta g(x) = 0$, so $\alpha f + \beta g \in M^a$

Closed subspace: Let $(f_i) \in M^a$ and $f_i \to f$ in X^*, then $f \in M^a$

Indeed, for any $x \in M$, $f(x) = \lim f_i(x) = 0$

$X^a = \{0\}$ and $\{0\}^a = X^*$

convergence in X^* is pointwise conv

3.10.3 Let $T : H \to H$ be bounded, self-adjoint, i.e. $T = T^*$ and there exists c s.t. $\|Tx\| \leq c\|x\|$ for all x.

Then $\|T^n x\| \leq c\|T^{n-1}x\| \leq \cdots \leq c^n \|x\|$, so T^n is bounded

Moreover $(T^n)^* = (T \cdots T)^* = T^* \cdots T^* = T \cdots T \cdot T^n$,

so T^n is self-adjoint

$(AB)^* = B^* A^*$

3.10.4 Let $T : H \to H$ be bounded.

$T_1^* = \left(\frac{1}{2}(T + T^*) \right)^* = \frac{1}{2}(T^* + T) = T_1$,

$T_2^* = \left(\frac{1}{2i}(T - T^*) \right)^* = -\frac{1}{2i}(T^* - T) = T_2$

Suppose $T_1 + iT_2 = T = S_1 + iS_2$ for self-adjoint operators S_1, S_2

Then $(T_1 - S_1) + i(T_2 - S_2) = 0$ and $\left[(T_1 - S_1) + i(T_2 - S_2) \right]^* = 0$,

i.e. $(T_1 - S_1) - i(T_2 - S_2) = 0$, hence $S_1 = T_1$ and $S_2 = T_2$.