9.8.1 Define $F_\lambda = \lim_{\mu \to \lambda^-} E_\mu$ (note that this limit exists by Thm 9.6.3 regarding monotone sequence of projections), then $\lambda < \mu$ implies that $E_\lambda \le E_\mu$, so $E_{\lambda - \frac{1}{n}} \le E_{\mu - \frac{1}{n}}$ for all n, i.e. $F_\lambda \le F_\mu$.

9.8.2 Set $\tilde{E}_\lambda = \lim_{\mu \to \lambda^+} E_\mu$ (Thm 9.6.3 also gives existence of limit in the case of monotone decreasing sequences), then $\lambda < \mu$ implies that $E_\lambda \le E_\mu$, so $E_{\lambda + \frac{1}{n}} \le E_{\mu + \frac{1}{n}}$ for all n, i.e. $\tilde{E}_\lambda \le \tilde{E}_\mu$.

9.8.3 Let $B = (T^2)^{1/2}$ and $T^+ = \frac{1}{2}(B+T)$, $T^- = \frac{1}{2}(B-T)$ and note that $BT = TB$ (by Thm. 9.4.2). Hence $T^+T^- = T^-T^+$, indeed

$$T^+T^- = \frac{1}{2}\cdot\frac{1}{2}(B^2 + TB - BT - T^2) = \frac{1}{2}\cdot\frac{1}{2}(B^2 + BT - TB - T^2) = T^-T^+.$$

9.8.4 Let $T = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$, then $T^2 = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}$ and all matrices $\begin{pmatrix} \pm 2 & 0 \\ 0 & \pm 3 \end{pmatrix}$ gives T^2 when squared, but $B = (T^2)^{1/2} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ is the positive square root ($\sigma(B) = \{2,3\}$). Moreover,

$$T^+ = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } T^- = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}$$

Extra 1: Let $A = C_0(X) = C^*(p_1, p_2, \dots)$ and let $x = \sum_{n=1}^{\infty} \frac{1}{3^n} p_n$

If $x_k = \sum_{n=1}^{k} \frac{1}{3^n} p_n$, then (x_n) is a Cauchy sequence in A, since $\| x_k - x_m \| = \| \sum_{n=m+1}^{k} \frac{1}{3^n} p_n \| \le \sum_{n=m+1}^{\infty} \frac{1}{3^n} = \frac{1}{2 \cdot 3^m} \longrightarrow 0$ as $m \to \infty$, hence $x \in A$ since A is complete.

Let $x, p_1, p_2, \dots$ be represented by functions $f, \chi_1, \chi_2, \dots$ on X, where χ_i is characteristic function on some (discrete) set $A_i \subset X$, which is the case since χ_i is a projection (i.e. $\chi_i(a) = \overline{\chi_i(a)} = \chi_i(a)^2$ for all a).

Note that $a \in A_1 \Rightarrow |f(a)| \ge \frac{1}{3}$ and $a \notin A_1 \Rightarrow |f(a)| \le \frac{1}{6}$, so we can define a cont function g on $\mathbb{R} \supset \sigma(x)$ since x is self-adjoint, with the property that $g = 0$ on $(-\infty, \frac{1}{6}]$ and $g = 1$ on $[\frac{1}{3}, \infty)$

This means that $a \in A_1 \Rightarrow (g \circ f)(a) = 1$ and $a \notin A_1 \Rightarrow (g \circ f)(a) = 0$, i.e. $g \circ f = \chi_1$, hence $g(x) = p_1$, hence $p_1 \in C^*(x)$ (check Murphy's book C^*-algebras and operator theory, section 2.1, page ~ 41-43).

Continue inductively to get that $p_2, p_3, \dots \in C^*(x)$ also

Extra 2: Suppose E^0 is finite, show that $\sum_{v \in E^0} p_v$ is a unit for $C^*(E)$

Obviously $p_u \sum_{v \in E_0} p_v = p_u^2 = p_u$ and $\left(\sum_{v \in E_0} p_v\right) p_u = p_u^2 = p_u$ for all $u \in E^0$.

Moreover $\left(\sum_{v \in E^0} p_v\right) s_e = \left(\sum_{v} \sum_{r(f) = v} s_f s_f^*\right) s_e s_e^* s_e = s_e s_e^* s_e s_e^* s_e = s_e$

(as $s_f s_f^* s_e s_e^* = 0$ if $e \neq f$)

and $s_e \left(\sum_{v \in E^0} p_v\right) = s_e s_e^* s_e \left(\sum_{v \in E^0} p_v\right) = s_e\, p_{s(e)} \cdot p_{s(e)} = s_e s_e^* s_e = s_e$