9.8.1 Define \( F_x = \lim_{\mu \to x} E_{\mu} \) (note that this limit exists by Thm 9.6.3 regarding monotone sequence of projections), then \( \lambda \mu n \) implies that \( E_{\lambda} \leq E_{\mu} \), so \( E_{\lambda^n} \leq E_{\mu^n} \) for all \( n \), i.e., \( E_{\lambda} \leq E_{\mu} \).

9.8.2 Set \( \bar{E}_x = \lim_{\mu \to x} E_{\mu} \) (Thm 9.6.3 also gives existence of limit in the case of monotone decreasing sequences), then \( \lambda \mu n \) implies that \( E_\lambda \leq E_{\mu} \), so \( E_{\lambda^n} \leq E_{\mu^n} \) for all \( n \), i.e., \( \bar{E}_\lambda \leq \bar{E}_\mu \).

9.8.3 Let \( B = (T_3)^2 \) and \( T = \frac{1}{2} (B + T) \), \( T = \frac{1}{2} (B - T) \), and note that \( BT = TB \) (by Thm 9.14.2). Hence \( T^* T^* = T - T^* \), indeed

\[
T^* T^* = \frac{1}{4} \frac{1}{2} \left( B^2 + TB - BT - T^2 \right) = \frac{1}{4} \frac{1}{2} \left( B^2 + BT - TB - T^2 \right) = T - T^*. 
\]

9.8.4 Let \( T = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix} \), then \( T^2 = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix} \) and all where \( \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix} \) gives \( T^2 \) when squared, but \( B = (T_3)^2 = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \) is the positive square root \( (\sigma(B) = \{ 0, 3 \}). \) Moreover,

\[
T^2 = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \text{ and } T^* = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}
\]

**Extra 1:** Let \( A = C_0(X) = \mathcal{C}^0 \left( \mathbb{R}, E_1 \right) \) and let \( x = \sum_i \frac{1}{3^n} \mu_n \)

If \( x_n = \sum_i \frac{1}{3^n} \mu_n \), then \( (x_n) \) is a sequence in \( A \), since \( \| x_n - x_m \| = \| \sum_i \frac{1}{3^n} \mu_n \| \leq \sum_i \frac{1}{3^n} = \frac{1}{2} 2^n \rightarrow 0 \) as \( n \to \infty \), hence \( x \in A \) since \( A \) is complete.

Let \( x, \mu_1, \mu_2, ... \) be represented by functions \( f, \chi_1, \chi_2, \ldots \) on \( X \), where \( \chi_i \) is characteristic function on some (discrete) set \( A_i \subset X \), which is the case since \( \chi_i \) is a projection (i.e., \( \chi_i(a) = \chi_i(a) \cdot X_i(a) \) for all \( a \)).

Note that \( 0 < A \) \( \Rightarrow f(a) \geq \frac{1}{2} \) and \( A \notin \mathbb{N} \) \( \Rightarrow f(a) \leq \frac{1}{2} \), so we can define a function \( g \) on \( \mathbb{R} \) \( \sigma(x) \) since \( x \) is self-adjoint, with the property that \( g = 0 \) on \( [-1, 0, 0] \) and \( g \geq 0 \) on \( \left[ \frac{1}{2}, 1 \right] \).
This means that $a \in A_1 \Rightarrow (g \circ f)(a) = 1$ and $a \notin A_1 \Rightarrow (g \circ f)(a) = 0$, i.e., $g \circ f : X \to Y$, hence $g \in C(X)$. (Check Murphy's book *C*-algebras and operator theory*, section 7.1, page 41-42).

Continue inductively to get that $p_1, p_2, \ldots \in C^*(X)$ also.

**Extra 2:** Suppose $E^0$ is finite, show that $\sum_{v \in E_0^*} p_v$ is a unit for $C^*(E)$.

 Obviously $p_u \sum_{v \in E_0^*} p_v = p_u^2 = p_u$ and $(\sum_{v \in E_0^*} p_v)^2 = p_u^2 = p_v$ for all $u \in E^0$.

 Moreover $(\sum_{v \in E_0^*} p_v)^* e_v = (\sum_{v \in E_0^*} p_v^*) e_v = \sum_{v \in E_0^*} p_v^* e_v = e_v$

for all $s \in E^0$, $s^* e_v = 0$ if $v \notin E$.

 and $s_e (\sum_{v \in E_0^*} p_v) = s_e s_v e_v (\sum_{v \in E_0^*} p_v) = s_e p_v(s_e) p_v(s_e) = s_e s_v s_e = s_e$