9.4.8 + 9

Let B be a nonsingular n-rowed real square matrix and $C = BB^T$.

Show that C has a nonsingular positive square root A and show that $D = A^{-1}B$ is orthogonal.

Proof: B is real so $B^* = B^T$.
It follows that $C = BB^T \geq 0$ so it has a positive square root A.

Likewise $C^{-1} = (B^TB)^{-1} = B^{-1}(B^T)^{-1} = B^{-1}(B^{-1})^T \geq 0$
So it has a positive square root A'.

C and C^{-1} commute so A and A' also commute. Hence $AA' \geq 0$ and

$$(AA')^2 = A^2(A')^2 = C^2(C^{-1})^2 = I$$

So AA' is a positive square root of I and is hence equal to I.

Likewise $A'A \geq 0$ and $(A'A)^2 = (A')^2A^2 = (C^{-1})^2C^2 = I$ which shows that $A'A = I$. Thus A is nonsingular and $A^{-1} = A'$.

We have $(A^T)^2 = (A^2)^T = C^T = (BB^T)^T = BB^T = C$
So A^T is a positive square root of C.
It follows that $A^T = A$.

Thus $D^T D = (A^{-1}B)^T (A^{-1}B) = B^T A^{-1} A^{-1} B = B^T C^{-1} B$
$= B^T (BB^T)^{-1} B = B^T (B^T)^{-1} B^{-1} B = I$

and $DD^T = (A^{-1}B)(A^{-1}B)^T = A^{-1} B B^T A^{-1} = A^{-1} A^2 A^{-1} = I$

Which shows that D is an orthogonal matrix.

9.5.1 Let P be a projection. Then $P = P^*$ and $\sigma(P) \subseteq \{0,1\}$ so $\langle Px, x\rangle \in [0,1]$ if $\|x\| = 1$. It follows that $0 \le \langle Px, x\rangle \le \langle x, x\rangle$ for all $x \in H$ which shows that $0 \le P \le I$.

$P = 0$ if and only $P(H) = \{0\}$, and $P = I$ if and only if $P(H) = H$.

9.5.3 Define $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x,y) = (x-y, 0)$. Then T is linear and $T^2(x,y) = T(x-y, 0) = (x-y, 0) = T(x,y)$ for all $(x,y) \in \mathbb{R}^2$ so $T^2 = T$.

If $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ then

$$\langle T(x_1, y_1), (x_2, y_2) \rangle = \langle (x_1 - y_1, 0), (x_2, y_2) \rangle$$
$$= (x_1 - y_1) x_2$$

and $\langle (x_1, y_1), T(x_2, y_2) \rangle = \langle (x_1, y_1), (x_2 - y_2, 0) \rangle$
$$= x_1 (x_2 - y_2)$$

so $T \neq T^*$.

Alternative solution: T has the matrix representation $\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$. $\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$
$= \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} \neq \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$

so $T^2 = T$ and $T^* = T$.

9.6.10 P_1 is a projection, $Y = P_1(H)$. and $P_1T = T$
Show that $T(Y) \subseteq Y$ and $T(Y^\perp) \subseteq Y^\perp$.

If $x \in H$ then $TP_1x = P_1Tx \in Y$ so $T(Y) \subseteq Y$.

If $y \in Y^\perp$ then $\langle y, P_1x \rangle = 0$ for all $x \in H$ and then

$\langle Ty, P_1x \rangle = \langle y, T^*P_1x \rangle = \langle y, P_1T^*x \rangle = 0$ for all $x \in H$, so $Ty \in Y^\perp$. This shows that $T(Y^\perp) \subseteq Y^\perp$.

$$\boxed{T^*P_1 = (P_1T)^* = (TP_1)^* = P_1T^*}$$

9.6.12

Show that if $T(Y) \subseteq Y$ and $T(Y^\perp) \subseteq Y^\perp$ then

$TP_1 = P_1T$. Let $x \in X$. Then

$P_1Tx = P_1T(P_1x + (\underline{x - P_1x})) = TP_1x$ because

$TP_1x \in Y$ and $T(\underline{x - P_1x}) \in Y^\perp$ so $P_1TP_1x = TP_1x$ and $P_1T(x - P_1x) = 0$.

$\mathcal{N}(P_1)$

9.6.13 Let P_2 be the projection onto $Y^\perp$.
Then $P_2 = I - P_1$ so $TP_2 = T - TP_1 = T - P_1T = P_2T$.

(Or Y reduces $T \Leftrightarrow T(Y) \subseteq Y \wedge T(Y^\perp) \subseteq Y^\perp \Leftrightarrow$
$Y^\perp$ reduces $T \Leftrightarrow TP_2 = P_2T$.)