9.1.6 $T: \ell^2 \to \ell^2$, $(x_1, x_2, x_3, \dots) \mapsto (0, 0, x_1, x_2, x_3, \dots)$

Note that $\|Tx\|^2 = \sum_1^\infty |x_i|^2 = \|x\|^2$ for all $x \in \ell^2$, so T is bounded.

Have that $(x, T^* e_i) = (Tx, e_i) = x_{i-2}$, $i \geq 3$, 0 else,

so $T^* e_i = e_{i-2}$, i.e. $T^*: (x_1, x_2, x_3, \dots) \mapsto (x_3, x_4, x_5, \dots)$,

so $T \neq T^*$. If $S: (x_1, x_2, x_3, \dots) \mapsto (0, x_1, x_2, x_3, \dots)$,

then $T = S^2$. (so square roots can exist even if T is not self-adjoint)

9.2.9 Let $A = A^*$ (Hermitian matrix), then $m = \inf_{\|x\|=1} \langle Ax, x\rangle = \inf_{x \neq 0} \frac{\langle Ax, x\rangle}{\langle x, x\rangle}$

$= \min_x q(x) = \lambda_n$, and similarly $M = \lambda_1$. since by 9.2.1 and 3,

m and M are the smallest and biggest spectral values of A

If A_j is the restriction of A to Y_j, then Y_j consists of

all eigenvectors corresponding to eigenvalues $\lambda_j, \dots, \lambda_n$ (theorem 9.1.1),

so $\lambda_j, \dots, \lambda_n$ are eigenvalues of A_j. The same argument as above

gives that $\lambda_j = \sup_{x \neq 0} \frac{\langle A_j x, x\rangle}{\langle x, x\rangle} = \sup_{\substack{x \neq 0 \\ x \in Y_j}} \frac{\langle Ax, x\rangle}{\langle x, x\rangle} = \max_{\substack{x \neq 0 \\ x \in Y_j}} q(x)$.

9.3.2 The ordering $\leq$ defined by $T_1 \leq T_2 \iff \langle T_1 x, x\rangle \leq \langle T_2 x, x\rangle$ for all x

is a partial order on the set of bounded self-adjoint operators:

(i) $T \leq T$ for all T (obvious)

(iii) $T_1 \leq T_2$ and $T_2 \leq T_3 \Rightarrow \langle T_1 x, x\rangle \leq \langle T_2 x, x\rangle \leq \langle T_3 x, x\rangle$ for all x,

so $T_1 \leq T_3$.

(ii) $T_1 \leq T_2$ and $T_2 \leq T_1$, then $\langle T_1 x, x\rangle = \langle T_2 x, x\rangle$ for all x,

hence $\langle (T_1 - T_2)x, x\rangle = 0$ for all x, i.e. $\|T_1 - T_2\| = 0$ so $T_1 = T_2$

(a) $T_1 \leq T_2 \Rightarrow \langle T_1 x, x\rangle \leq \langle T_2 x, x\rangle \Rightarrow \langle T_1 x, x\rangle + \langle Tx, x\rangle \leq \langle T_2 x, x\rangle + \langle Tx, x\rangle$

(b) $T_1 \leq T_2 \Rightarrow \langle T_1 x, x\rangle \leq \langle T_2 x, x\rangle \Rightarrow \alpha\langle T_1 x, x\rangle \leq \alpha\langle T_2 x, x\rangle$, $\alpha \geq 0$

(The set of positive elements is a closed cone in $B(H)$)

9.3.9 If $T \geq 0$, then $\langle Tx, x \rangle \geq 0$ for all x, hence $m = \inf_{\|x\|=1} \langle Tx, x \rangle \geq 0$,

meaning that $\sigma(T) \subset [0, \infty)$ by theorem 9.2.3.

Thus $-1 \notin \sigma(T)$ so $T + I$ is invertible

9.3.10 Note that $\langle T^*Tx, x \rangle = \langle Tx, Tx \rangle = \|Tx\|^2 \geq 0$ for all x,

so T^*T is positive.

9.3.11 Let $P_n : \ell^2 \to \ell^2$ be given by $(x_1, x_2, \dots) \mapsto (x_1, x_2, \dots, x_n, 0, 0, \dots)$

Then $m \leq n \Rightarrow P_m \leq P_n$ since $\langle (P_n - P_m)x, x \rangle = |x_{m+1}|^2 + \dots + |x_n|^2 \geq 0$,

and $P_n \leq I$ for all n since $\langle (I - P_n)x, x \rangle = |x_{n+1}|^2 + \dots \geq 0$.

Moreover $P_m P_n = P_n P_m = P_{\min\{m,n\}}$

Then $\lim_{n \to \infty} \| Ix - P_n x \|^2 = |x_{n+1}|^2 + \dots \to 0$