7.3.9 $T: \ell^\infty \to \ell^\infty, \quad (x_1, x_2, x_3, \dots) \mapsto (x_2, x_3, x_4, \dots)$

$\|T\| = \sup_{\|x\|=1} \|Tx\| = 1 \quad$ ($\|Tx\| \le \|x\|$ obviously and $\|Te_2\| = 1$)

a) $\lambda \in \sigma(T) \Rightarrow |\lambda| \le \|T\| = 1$ by Thm 7.3.4, so $|\lambda| > 1 \Rightarrow \lambda \in \sigma(T)^c = \rho(T)$

b) Let $|\lambda| \le 1$ and suppose $Tx = (x_2, x_3, x_4, \dots) = (\lambda x_1, \lambda x_2, \lambda x_3, \dots) = \lambda x$,

then $x_{i+1} = \lambda x_i$ for all i, so if $x_1 = 1$, then $x = (1, \lambda, \lambda^2, \dots)$,

and $Y = \operatorname{span} x$

7.4.8 i)

$$\det A - \lambda I = \begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)(\lambda^2 - 1) = 0 \Leftrightarrow \lambda = \pm 1$$

ii) $A^2 = I$, hence $\sigma(A)^2 = \sigma(I) = \{1\}$, so $\lambda \in \sigma(A) \Leftrightarrow \lambda^2 = 1$, i.e. $\lambda = \pm 1$

7.4.9 Examples (projections): $T: \ell^\infty \to \ell^\infty, \quad (x_1, x_2, \dots) \mapsto (x_1, x_2, \dots, x_n, 0, 0, \dots)$

$X = C_0((0,1) \cup (1,2)), \ T: X \to X, \quad Tf(x) = \begin{cases} f(x), & x \in (0,1) \\ 0, & x \in (1,2) \end{cases}$

a) Formula (9) gives for $|\lambda| > \|T\| = 1$ that

$$(T - \lambda I)^{-1} = -\frac{1}{\lambda} \sum_{j=0}^{\infty} \left(\frac{1}{\lambda} T\right)^j = -\frac{1}{\lambda} I - \frac{1}{\lambda} T \sum_{j=1}^{\infty} \frac{1}{\lambda^j} = -\frac{1}{\lambda} I - \frac{1}{\lambda(\lambda - 1)} T$$

Formula also holds for all other $\lambda \ne 0, 1$.

b) Let $p(x) = x^2 - x$, so $p(T) = 0$. Then $p(\sigma(T)) = \sigma(p(T)) = \{0\}$,

so $\lambda \in \sigma(T) \Rightarrow \lambda^2 = \lambda$ i.e. $\lambda = 0$ or 1. $\sigma(T) = \{0\} \Rightarrow T = 0$ and $\sigma(T) = 1 \Rightarrow \sigma(I - T) = \{0\}$

i.e. $I - T = 0$

7.5.5 Note that $\|(ST)^n\| = \|S^n T^n\| \le \|S^n\| \cdot \|T^n\|$ for all n,

hence $\|(ST)^n\|^{\frac{1}{n}} \le \|S^n\|^{\frac{1}{n}} \|T^n\|^{\frac{1}{n}}$ for all n, thus when approaching the limit

$r_\sigma(ST) \le r_\sigma(S) \cdot r_\sigma(T)$

7.5.9 Suppose $T = T^*$ first, then $\|T^{2^n}\| = \|(T^{2^{n-1}})^* T^{2^{n-1}}\| = \|T^{2^{n-1}}\|^2 = \dots = \|T\|^{2^n}$,

so $r_\sigma(T) = \lim_{n\to\infty} \|T^{2^n}\|^{1/2^n} = \|T\|$

Suppose now $T^*T = TT^*$, then

$\|T^{2^n}\| = \|(T^{2^n})^* T^{2^n}\|^{1/2} = \|(T^*T)^{2^n}\|^{1/2} = \|T^*T\|^{2^n/2} = \|T\|^{2^n}$ again

7.6.3 Let $\tilde{c}_0$ be the space of sequences with only finitely many nonzero terms.

We need to complete the space and define multiplication.

Let this space be c_0 consisting of all sequences converging to zero,

equipped with sup-norm and pointwise multiplication

(c_0 does not have a unit, but c (all convergent sequences) is a unital example)

7.7.7 Let $x \in A$, $x \neq 0$, $\lambda \in \sigma(x) \neq \emptyset$, then $x - \lambda e$ is not invertible

Hence $x - \lambda e$ (since A is a division ring), so $x = \lambda e$

(illustrates importance of Thm. 7.7.4)