7.3.4 $T : \ell^2 \to \ell^2$, $(x_i) \mapsto (\alpha_i x_i)$, (α_i) dense in $[0,1]$.

Then $\lambda \in \sigma_p(T)$ if and only if $Tx = \lambda x$ for some $x \neq 0$,

i.e. $(\alpha_i x_i) = (\lambda x_i)$ for some $x \neq 0$. This holds if $\lambda = \alpha_i$ for some i (and $x = e_i$)

Thus $\sigma_p(T) = (\alpha_i)$. Since $\sigma(T)$ is closed, $[0,1] \subset \sigma(T)$

If $\mu \notin [0,1]$, then $T - \mu I$ has bounded inverse, namely $(x_i) \longmapsto \left(\frac{1}{\alpha_i - \mu} x_i\right)$

(as $\inf_i |\alpha_i - \mu| > 0$) Hence $\sigma(T) = [0,1]$.

7.3.5 Let $\lambda \in \sigma(T) \setminus \sigma_p(T)$, then $(T - \lambda I)^{-1} : (x_i) \longmapsto \left(\frac{1}{(\alpha_i - \lambda)} x_i\right)$

For each $N > 0$ there exists i s.t. $|\alpha_i - \lambda| < \frac{1}{N}$ so that

$\|(T-\lambda I)^{-1}\| = \sup_{\|x\|=1} \|(T-\lambda I)^{-1} x\| \geq \|(T-\lambda I)^{-1} e_i\| = \left|\frac{1}{\alpha_i - \lambda}\right| > N.$

hence $(T - \lambda I)^{-1}$ is unbounded

(alternatively, pick a subsequence of (α_i) converging to λ).

7.3.6 Since K is separable, there exists a countable dense subset of K

Write this set as a sequence (α_i) and proceed as above.

Note that $(x_i) \mapsto (\alpha_i x_i)$ is bounded as K is compact.

7.4.4 Note that if $p(T)x = y$ has unique solution x for each $y \in X$,

then $p(T)$ is bijective. $p(T)$ is also bounded, hence it has a bounded

inverse $p(T)^{-1}$ (by OMT). This means $0 \notin \sigma(p(T)) = p(\sigma(T))$,

so $p(\lambda) \neq 0$ for all $\lambda \in \sigma(T)$.

For the opposite direction, assume $p(\lambda) \neq 0$ for all $\lambda \in \sigma(T)$.

$0 \notin p(\sigma(T)) = \sigma(p(T))$, hence $p(T)$ has a bounded inverse,

in particular, it is bijective.

7.5.1 Since $\sigma(T)^m = \sigma(T^m) = \{0\}$, then $\sigma(T) = \{\lambda : \lambda^m = 0\} = \{0\}$

7.7.4 $\left\{ \begin{bmatrix} \alpha & \beta \\ 0 & 0 \end{bmatrix} : \alpha, \beta \in \mathbb{C} \right\}$ is a subalgebra of $M_2(\mathbb{C})$, call it A.

The only thing to check is that $xy \in A$ if $x, y \in A$, indeed:

$$\begin{bmatrix} \alpha & \beta \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \alpha' & \beta' \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \alpha\alpha' & \alpha\beta' \\ 0 & 0 \end{bmatrix}.$$

Note that A does not have an identity so we calculate $\sigma(T)$ in $M_2(\mathbb{C})$

$$\det(x - \lambda I) = \begin{vmatrix} \alpha - \lambda & \beta \\ 0 & -\lambda \end{vmatrix} = \lambda(\lambda - \alpha) = 0 \iff \lambda = 0 \text{ or } \alpha,$$

so $\sigma(x) = \{0, \alpha\}$

7.7.5 Let $x \in B \subset A$, and suppose $\lambda \notin \sigma_B(x)$. Then $x - \lambda$ has an inverse $z \in B$, i.e. $z(x-\lambda) = (x-\lambda)z = e$. But $z \in A$ also, so $\lambda \notin \sigma_A(x)$, hence $\sigma_A(x) \subset \sigma_B(x)$