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Introduction to C^*-algebras

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C^*-algebras

Let H be a complex Hilbert space. Then $B(H, H)$ is a complex Banach algebra. The map $T \mapsto T^*$ is an *involution* on $B(H, H)$, i.e., it is a map from $B(H, H)$ to $B(H, H)$ satisfying:

1. $(T^*)^* = T$
2. $(T + S)^* = T^* + S^*$
3. $(\alpha T)^* = \overline{\alpha} T^*$
4. $(TS)^* = S^* T^*$

for $T, S \in B(H, H)$ and $\alpha \in \mathbb{C}$.
We furthemore have that $\|T^* T\| = \|T\|^2$ for all $T \in B(H, H)$.

Definition

A *C^*-algebra* is a complex Banach algebra A with an involution satisfying $\|x^* x\| = \|x\|^2$ for all $x \in A$.

Examples of C^*-algebras

- Let H be a complex Hilbert space and A a closed $*$-subalgebra of $B(H, H)$, i.e., A is a closed subset of $B(H, H)$ such that the following hold for $T, S \in A$ and $\alpha \in \mathbb{C}$:
  1. $T + S \in A$,
  2. $\alpha T \in A$,
  3. $TS \in A$,
  4. $T^* \in A$.

  Then A is a C^*-algebra.
- Let $M_n(\mathbb{C})$ denote the set of complex $n \times n$-matrices. Then $M_n(\mathbb{C})$ is a C^*-algebra.

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Examples of C^*-algebras

- Let X be a locally compact Hausdorff space. Then

$$C_0(X) = \{f : X \to \mathbb{C} \mid f \text{ is continuous and } \{x \in X \mid |f(x)| \geq \epsilon\} \text{ is compact for all } \epsilon > 0\}$$

is a C^*-algebra with f^* defined by $f^*(x) = \overline{f(x)}$.

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C^*-subalgebras

Let A be a C^*-algebra. A closed subset B of A is called a *C^*-subalgebra* of A if the following holds for $x, y \in B$ and $\alpha \in \mathbb{C}$:

1. $x + y \in B$,
2. $\alpha x \in B$,
3. $xy \in B$,
4. $x^* \in B$.

If X is a subset of A, then $C^*(X)$ will denote the smallest C^*-subalgebra of A which contains X.

$*$-homomorphisms

Let A and B be C^*-algebras. A map $\phi : A \to B$ is a $*$-homomorphism if

1. $\phi(x+y) = \phi(x) + \phi(y)$
2. $\phi(\alpha x) = \alpha\phi(x)$
3. $\phi(xy) = \phi(x)\phi(y)$
4. $\phi(x^*) = \phi(x)^*$

for $x, y \in A$ and $\alpha \in \mathbb{C}$.

Proposition

Let A be a unital C^*-algebra and let $x \in A$ be self-adjoint (i.e, $x^* = x$). Then the spectral radius of x is equal to $\|x\|$.

It follows from this proposition that if ϕ is a $*$-homomorphism, then $\|\phi(x)\| \leq \|x\|$ for all $x \in A$.

The Gelfand-Naimark theorem

- If $\phi : A \to B$ is a $*$-homomorphism, then $\phi(A)$ is a C^*-subalgebra of B.
- A bijective $*$-homomorphism is called a $*$-isomorphism. Notice that if ϕ is a $*$-isomorphism, then $\|\phi(x)\| = \|x\|$ for all $x \in A$.
- A $*$-homomorphism is injective if and only if it is an isometry (i.e., $\|\phi(x)\| = \|x\|$ for all $x \in A$).

Theorem

Let A be a C^*-algebra. Then there exists a complex Hilbert space H and an injective $*$-homomorphism $\phi : A \to B(H, H)$.

The Gelfand theorem

Theorem

Let A be a commutative C^*-algebra (i.e., $xy = yx$ for all $x, y \in A$). Then there exists a locally compact Hausdorff space X and a $*$-isomorphism $\phi : A \to C_0(X)$.
If A is unital, then X is compact and $C_0(X) = C(X)$.

Functional calculus for normal elements

Let A be a C^*-algebra. An element $x \in A$ is called *normal* if $xx^* = x^*x$.

Theorem

Let A be a unital C^*-algebra and $x \in A$ a normal element. Then there is a $*$-isomorphism $\phi : C(\sigma(x)) \to C^*(1, x)$ such that

1. $\phi(1) = 1$,
2. $\phi(\mathrm{id}) = x$, and
3. $\phi(\{f \in C(\sigma(x)) \mid f(0) = 0\}) = C^*(x)$.

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Partial isometries

Let H be a complex Hilbert space. An operator $T \in B(H, H)$ is called a *partial isometry* if $\|Tx\| = \|x\|$ for all $x \in \mathcal{N}(T)^{\perp}$.

Proposition

TFAE

1. T is a partial isometry,
2. T^* is a partial isometry,
3. TT^* is a projection,
4. T^*T is a projection,
5. $TT^*T^* = T$,
6. $T^*TT^* = T^*$.

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Positiv elements, projections and partial isometries in C^*-algebras

Let A be a C^*-algebra. Then

- $a \in A$ is called *positive* if $a = bb^*$ for some $b \in A$,
- $p \in A$ is called a *projection* if $p^2 = p^* = p$,
- $s \in A$ is called a *partial isometry* if $ss^*s = s$.

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Finite and infinite projections

Let A be a C^*-algebra. We say that two projections $p, q \in A$ are (Murray-von Neumann) equivalent if there is a partial isometry $s \in A$ such that $ss^* = p$ and $s^*s = q$.
We say that a projection $p \in A$ is

- *infinite* if there is a projection $q \in A$ such that $p \sim q$, $q \leq p$ and $q \neq p$,
- *finite* if $q \sim p$, $q \leq p$ implies $q = p$,
- *properly infinite* if there are projections $p_1, p_2 \in A$ such that $p \sim p_1$, $p \sim p_2$, $p_1 \leq p$, $p_2 \leq p$ and $p_1 \perp p_2$.

Examples

- Let H be a complex Hilbert space. Then a projection $P \in B(H, H)$ is (properly) infinite if and only if $P(H)$ is infinite dimensional.
- Let X be a locally compact Hausdorff space. Then every projection $p \in C_0(X)$ is finite.

Directed graphs

A *(directed) graph* $E = (E^0, E^1, r, s)$ consists of two sets E^0 and E^1 and two functions $r, s : E^1 \to E^0$.

The elements of E^0 are called *vertices* and the elements of E^1 are called *edges*.

For each each e, $s(e)$ is called the *source* of e and $r(e)$ the *range* of e. If $s(e) = v$ and $r(e) = w$, then we say that v *emits* e and that w *receives* e, or that e is an edge from v to w.

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Graph C^*-algebras

Let $E = (E^0, E^1, r, s)$ be a graph. The C^*-algebra of E is the universal C^*-algebra $C^*(E)$ generated by a family of mutually orthogonal projections $\{p_v \mid v \in E^0\}$ and a family of partial isometries $\{s_e \mid e \in E^1\}$ with mutually orthogonal ranges satisfying

1. $s_e^* s_e = p_{s(e)}$ for all $e \in E^1$,
2. $s_e s_e^* \leq p_{r(e)}$ for all $e \in E^1$, and
3. $p_v = \sum_{e \in r^{-1}(v)} s_e s_e^*$ for every $v \in E^0$ such that $0 < |r^{-1}(v)| < \infty$.

I.e, $C^*(E) = C^*(\{p_v \mid v \in E^0\} \cup \{s_e \mid e \in E^1\})$ where $\{p_v \mid v \in E^0\}$ is a family of mutually orthogonal projections and $\{s_e \mid e \in E^1\}$ is a family of partial isometries with mutually orthogonal ranges satisfying (1)–(3) above,

Graph C^*-algebras

and if B is another C^*-algebra with a family of mutually orthogonal projections $\{p'_v \mid v \in E^0\}$ and a family of partial isometries $\{s'_e \mid e \in E^1\}$ with mutually orthogonal ranges satisfying (1)–(3) above, then there exists a unique $*$-homomorphism $\phi : C^*(E) \to B$ such that $\phi(p_v) = p'_v$ for each $v \in E^0$ and $\phi(s_e) = s'_e$ for each $e \in E^1$.

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Ideals and quotients of C^*-algebras

Let A be a C^*-algebra. A (closed, two-sided) ideal is a closed subet $I \subseteq A$ such that the following hold for $x, y \in I$, $a \in A$ and $\alpha \in \mathbb{C}$:

1. $x + y \in I$,
2. $\alpha x \in I$,
3. $ax, xa \in I$.

- If I is an ideal, then $x \in I \implies x^* \in I$, so I is a C^*-algebra of A.
- If ϕ is a $*$-homomorphism, then $\mathcal{N}(\phi)$ is an ideal.
- If I is an ideal, then $A/I = \{a + I \mid a \in A\}$ is a C^*-algebra with norm $\|a + I\| = \inf\{\|a + x\| \mid x \in I\}$.

Toeplitz graph algebras

Let $E = (E^0, E^1, r, s)$ be a graph. The *Toeplitz* algebra of E is the universal C^*-algebra $\mathcal{T}(E)$ generated by a family of mutually orthogonal projections $\{q_v \mid v \in E^0\}$ and a family of partial isometries $\{t_e \mid e \in E^1\}$ with mutually orthogonal ranges satisfying

1. $t_e^* t_e = q_{s(e)}$ for all $e \in E^1$,
2. $t_e t_e^* \leq q_{r(e)}$ for all $e \in E^1$.

There is for any graph E a surjective $*$-homomorphism $\phi : \mathcal{T}(E) \to C^*(E)$ such that $\phi(q_v) = p_v$ for $v \in E^0$ and $\phi(t_e) = s_e$ for $e \in E^1$. The null space of ϕ is the ideal generated by $\{q_v - \sum_{e \in r^{-1}(v)} t_e t_e^* \mid v \in E^0,\ 0 < |r^{-1}(v)| < \infty\}$.

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Direct sums and products of C^*-algebras

- If $\{A_1, A_2, \ldots, A_n\}$ is a finite family of C^*-algebras, then the *direct sum* $A_1 \oplus \cdots \oplus A_n$ is a C^*-algebra with norm $\|(a_1, \ldots, a_n)\| = \max\{\|a_i\| \mid i \in \{1, \ldots, n\}\}$.
- If $\{A_i \mid i \in I\}$ is an infinite family of C^*-algebras, then the *direct product*

$$\prod_{i \in I} A_i = \{(a_i)_{i \in I} \mid \sup\{\|a_i\| \mid i \in I\} < \infty\}$$

is a C^*-algebra with norm $\|(a_i)_{i \in I}\| = \sup\{\|a_i\| \mid i \in I\}$, and the *direct sum*

$$\bigoplus_{i \in I} A_i = \{(a_i)_{i \in I} \mid \{i \in I \mid \|a_i\| > \epsilon\} \text{ is finite for all } \epsilon > 0\}$$

is an ideal in $\prod_{i \in I} A_i$.

Inductive limits of C^*-algebras

- An *inductive system* of C^*-algebras is a system $\{(A_i, \phi_{ij}) \mid i, j \in I,\ i \leq j\}$ where I is a directed set, each A_i is a C^*-algebra, and each ϕ_{ij} is a $*$-homomorphism from A_i to A_j such that $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$ for $i \leq k \leq j$.
- The *inductive (or direct) limit* $\varinjlim(A_i, \phi_{ij})$ of the inductive system $\{(A_i, \phi_{ij}) \mid i, j \in I,\ i \leq j\}$ is the universal C^*-algebra A for which there are $*$-homomorphisms $\phi_i : A_i \to A$ such that $\phi_i = \phi_j \circ \phi_{ij}$ for $i \leq j$ and A is the closure of $\bigcup_{i \in I} \phi_i(A_i)$.

Example

If A is a C^*-algebra, $A_1 \subseteq A_2 \subseteq \dots$ is an increasing sequence of C^*-subalgebras of A, and we for $i \leq j$ let ι_{ij} denote the inclusion of A_i into A_j, then the inductive limit $\varinjlim(A_i, \iota_{ij})$ is the closure of $\bigcup_{i \in \mathbb{N}} A_i$.

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