FUNCTIONAL ANALYSIS

Exercise 3

1) Here we discuss some concepts related to the category theorem.
   a) Let $A \subseteq B$ be two subsets of a normed space. Show that if B is of the first category then so is A.
   b) Is the closer of a nowhere dense set also nowhere dense? Is the closer of a set of the first category also of the first category? Is the interior of a set of the second category also of the second category?
   c) Let W be a proper subspace of a normed space V (i.e., $W \subset V$, $W \neq V$). Show that if W is closed then W is nowhere dense in V.
   d) Is C_{00} nowhere dense, of the first category or of the second category in l^∞? What about C_{00} in C_0?
   *e) Show that the set $\{f \in C[0,1] : \exists x_0 \in [0,1] \text{ s.t. } \exists f'(x_0)\}$ (the set of all functions which have a derivative in at least one point) is of the first category in $C[0,1]$.

2) Here we discuss properties of bounded linear operators. Let V and W be two normed spaces and let $T : V \to W$ be a linear operator.
   a) Show that if T is bounded then $\mathrm{Ker}T$ is a closed subspace of V.
   b) Is it true that if T is bounded then $\mathrm{Im}T = TV$ is a closed subspace of W?
   c) Show that if T^{-1} exists then $\|T\|^{-1} \leq \|T^{-1}\|$.
   d) Show that if T is bounded and $A \subseteq V$ is dense in V then $\|T\| = \sup_{x\in A} \frac{\|Tx\|}{\|x\|}$.
   e) Show that if T is an isomorphism then V is separable if and only if W is separable and V is a Banach space if and only if W is a Banach space.
   f) Show that if T is an isomorphism then for any $A \subseteq V$ we have $\overline{TA} = T\overline{A}$ and $(TA)^0 = T(A^0)$.

3) Here we introduce some specific linear operators and discuss their properties.
   a) Recall that for any $1 \leq p < \infty$ we define the $p-$norm on $\mathbb{R}^n$ by $\|\{x_n\}\|_p = (\sum_1^n |x_n|^p)^{1/p}$ and for $p = \infty$ by $\|\{x_n\}\|_\infty = \sup\{|x_n|\}$. Let $A \in M_n(\mathbb{R})$ and define an operator $A : \mathbb{R}^n \to \mathbb{R}^n$ by $x \to Ax$. Denote by $\|A\|_p$ the norm of the operator A when it acts from $(\mathbb{R}^n, \| - \|_p)$ to itself. Show that

$$\|A\|_\infty = \max_{1\leq i\leq n} \sum_{j=1}^n |A_{i,j}|,$$

and that

$$\|A\|_1 = \max_{1\leq j\leq n} \sum_{i=1}^n |A_{i,j}|.$$

b) For $k(x,y) \in C[0,1]^2$ define $T : C[0,1] \to C[0,1]$ by

$$(Tf)(x) = \int_0^1 k(x,y)f(y)dy. \tag{1}$$

Show that this operator is indeed into $C[0,1]$. Is it bounded?

c) For $k(x,y) \in L^2[0,1]^2$ define $T : L^2[0,1] \to L^2[0,1]$ by (1). Show that this operator is indeed into $L^2[0,1]$. Is it bounded?

d) Define the space of two sided sequences $l^q(\mathbb{Z}) = \{\{a_n\}_{-\infty}^{\infty} : \|\{a_n\}\|^q := \sum |a_n|^q \leq \infty\}$. For a sequence $\{a_n\}_{-\infty}^{\infty} \in l^1(\mathbb{Z})$ define an operator $T : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ by $\{b_n\}_{-\infty}^{\infty} \to \{\sum_m a_{m-n}b_m\}_{n=-\infty}^{\infty}$. Show that this operator is indeed into l^2. Is it bounded?

e) For $\{a_n\} \in l^\infty$ let $T : l^1 \to l^1$ be the operator defined by $\{b_n\} \to \{a_n b_n\}$. We saw in class that this operator is bounded and computed its norm. Under which conditions on $\{a_n\}$ will this operator be an isomorphism? In this case, what is $\|T^{-1}\|$? Under which conditions on $\{a_n\}$ will T be an isometry?

4) Here we discuss finite dimensional spaces.

a) Let V be a finite dimensional space over a field $\mathcal{F}$ (where $\mathcal{F}= \mathbb{R}$ or $\mathbb{C}$). Show that if the dimension of V is n, then V is isomorphic to $(\mathcal{F}^n, \| - \|_2)$. (A hint for this exercise appears below :).

b) Conclude that any two finite dimensional spaces over $\mathcal{F}$, with dimension n, are isomorphic.

c) Conclude that all finite dimensional spaces are Banach spaces.

d) Conclude that if W is a finite dimensional subspace of V, then W is closed.

d) Conclude that any two norms, $\| - \|_1$ and $\| - \|_2$, on a finite dimensional space V, are equivalent. I.e., there exist two positive constants c and C such that for every $x \in V$:

$$c\|x\|_1 \leq \|x\|_2 \leq C\|x\|_1.$$

Hint for exer. 4a: you might want to use known facts about the spaces $\mathcal{F}^n$, like the fact that in these spaces every bounded and closed set is compact.