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Department of Mathematical Sciences

Contact during the exam:
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Final Exam in TMA4225 Foundations of Analysis

English
Monday, December 1, 2008
Hours: 09.00–13.00
Aids: Code B – All printed and handwritten aids permitted. Approved calculator permitted.

Grades will be announced no later than: December 22, 2008.

Problem 1

a) Let $f_n(x) = \frac{1}{n}\chi_{[-n,n]}(x)$ for $x \in \mathbf{R}$, $n \in \mathbf{N}$. Show that f_n converges uniformly to 0 on $\mathbf{R}$.

b) Is there a Lebesgue integrable function g defined on $\mathbf{R}$ such that $f_n(x) \leq g(x)$ for all $x \in \mathbf{R}$, $n \in \mathbf{N}$? Justify your answer.

Problem 2 Let X be a set, and A and B two subsets of X. Explain why $|\chi_B - \chi_A|$ is the characteristic function of some subset $C \subset X$, and express C in terms of A and B.

Problem 3 Suppose f is nonnegative and integrable on $\mathbf{R}^d$. For each $y > 0$, let $E_y = \{x \in \mathbf{R}^d \colon f(x) > y\}$. Show that

$$\int_{\mathbf{R}^d} f(x)\,dx = \int_0^\infty m(E_y)\,dy\,.$$

Justify your steps, and indicate which theorems you are using.

[*Hint:* Consider the subgraph of f, given by $\{(x, y) \in \mathbf{R}^d \times \mathbf{R} \mid 0 < y < f(x)\}$.]

Problem 4 Let $f\colon [0,\infty) \to \mathbf{R}$ be defined by

$$f(t) = \begin{cases} \frac{\sin t}{t}, & t > 0 \\ 1, & t = 0\,. \end{cases}$$

a) Set $a_n = \int_{n\pi}^{(n+1)\pi} f(t)\,dt$ for $n \geq 0$. Prove that the series $\sum_{n=0}^{\infty} a_n$ converges by showing that: (i) The series is alternating, (ii) $|a_n| \geq |a_{n+1}|$ for $n \geq 0$, (iii) $\lim_{n\to\infty} a_n = 0$.
Use this to show that the improper Riemann integral $\int_{[0,\infty)}^{\mathcal{R}} f(t)\,dt \stackrel{\text{def}}{=} \lim_{b\to\infty} \int_0^b f(t)\,dt$ exists.

[*Hint:* Show that $\frac{2}{(n+1)\pi} \leq |a_n| \leq \frac{2}{n\pi}$ for $n \geq 1$.]

b) Show that the function f is *not* Lebesgue integrable over $[0,\infty)$.

Problem 5

a) Let $b > 0$ and define $f\colon [0,b] \to \mathbf{R}$ by

$$f(x) = \begin{cases} x^{-\frac{1}{2}}, & x \in (0,b] \\ 0, & x = 0\,. \end{cases}$$

Prove f is Lebesgue integrable, and compute the value of the Lebesgue integral $\int_0^b f(x)\,dx$.
[*Hint:* Consider the sequence $f_n = f\chi_{[\frac{1}{n},b]}$, and use the Monotone Convergence Theorem. You may use the Fundamental Theorem of Calculus to evaluate any Riemann integral you encounter along the way.]

b) Define $g\colon [0,1] \to \mathbf{R}$ by

$$g(x) = \begin{cases} x^{\frac{3}{2}} \sin(\frac{1}{x}), & x \in (0,1] \\ 0, & x = 0\,. \end{cases}$$

Show that g' is integrable on $[0,1]$ and that $g(x) = \int_0^x g'(t)\,dt$ for $x \in [0,1]$. Conclude that g is absolutely continuous, hence of bounded variation, on $[0,1]$.

[*Hint:* To prove $g(x) = \int_0^x g'(t)\,dt$, use an argument similar to that in **a)**, but this time in conjunction with the Dominated Convergence Theorem.]

c) Define $h\colon [0,1] \to \mathbf{R}$ by

$$h(x) = \begin{cases} x\sin(\frac{1}{x}), & x \in (0,1] \\ 0, & x = 0\,. \end{cases}$$

Show that h is *not* of bounded variation on $[0,1]$.

d) Consider the Cantor-Lebesgue function $F\colon [0,1] \to [0,1]$ as defined in Exercise 2(d) on page 38 in the textbook. You may use the following facts: F is a non-decreasing, uniformly continuous function which maps $[0,1]$ *onto* $[0,1]$; in addition, F it is differentiable a.e. with $F' = 0$ a.e.
Is F of bounded variation? Is F absolutely continuous? Justify your answers.