**Department of Mathematical Sciences**

**Problem 1.** Let

$$A = \bigcap_{n=1}^{\infty} \left( \bigcup_{m=1}^{\infty} \left[ \frac{1}{mn+1}, \infty \right) \right).$$

Find $m(A)$, where $m$ denotes the Lebesgue measure on $\mathbb{R}$.

**Solution:** Since

$$\bigcup_{m=1}^{\infty} \left[ \frac{1}{mn+1}, \infty \right) = (0, \infty)$$

for all $n \in \mathbb{N}$, we obtain that $A = (0, \infty)$. Hence, $m(A) = \infty$.

**Problem 2.**

Does exist an open set $A \subset (0, 1)$ with the following two properties:

1. $m(A) > 0$;
2. $A$ doesn't contain any binary rational?

**Solution:** There is no such a set.

Binary rationals from (0,1) have the form $m/2^n$, where $m, n \in \mathbb{N}$. Since $A$ is open and $m(A) > 0$, then $(a, b) \subset A$ for some $b > a$. Every such an interval contains a binary rational since binary rationals are dense in $\mathbb{R}$.

**Problem 3.**

Let

$$f(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} I_{[m^2,\, m^2+3^{-(m+n)}]}(x).$$

a. Prove that $f$ is Lebesgue measurable.

b. Find the Lebesgue integral

$$\int_{\mathbb{R}} f(x) dx.$$

**Solution:** Clearly, $f$ is Lebesgue measurable, since it is a countable sum of unsigned simple functions. By Tonelli's theorem

$$\int_{\mathbb{R}} f(x) dx = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \int_{\mathbb{R}} I_{[m^2,\, m^2+3^{-(m+n)}]}(x) dx = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} 3^{-(m+n)} = \sum_{m=1}^{\infty} 3^{-m} \sum_{n=1}^{\infty} 3^{-n} = 1/4.$$

**Problem 4.** a. Find the following limit

$$\lim_{n \to \infty} \int_1^{\infty} (\log x)^n e^{-x} dx.$$

**Solution:** All the functions are clearly unsigned measurable. Therefore by monotonicity of Lebesgue integral we get for each $n \in \mathbb{N}$

$$\int_1^{\infty} (\log x)^n e^{-x} dx \geq \int_3^4 (\log x)^n e^{-x} dx \geq (\log 3)^n e^{-4}.$$

Therefore,

$$\lim_{n \to \infty} \int_1^{\infty} (\log x)^n e^{-x} dx = \infty.$$

b. Prove that the function

$$f(t) = \int_1^\infty e^{-x^3} \cos xt \, dx$$

is continuous in each point $t \in \mathbb{R}$.

**Solution:**

Fix $t \in \mathbb{R}$ and take any sequence $t_n$ such that $\lim_{n\to\infty} t_n = t$. Denote

$$f_n(x) = e^{-x^3} \cos xt_n.$$

Since $|f_n(x)| \leq h(x)$ for all $x \in [1, \infty)$, where

$$h(x) = e^{-x^3},$$

and $h \in L_1([1, \infty))$, we get by Dominated Convergence Theorem that

$$\lim_{n\to\infty} f(t_n) = \lim_{n\to\infty} \int_1^\infty e^{-x^3} \cos xt_n dx = \int_{\mathbb{R}} e^{-x^3} \cos xt dx = f(t).$$

Since $t_n$ was arbitrary, this implies continuity of $f$ in $t$.

c. Find the following limit

$$\lim_{n\to\infty} \int_0^{n^{2/3}} (\log(1 + x/n) + \log(1 - x/n)) \; dx.$$

**Solution:**

All the functions are measurable since they are continuous on $[0, n^{2/3}]$ for $n > 1$. Also we have that when $x \to 0$,

$$\log(1 + x) = x - x^2/2 + O(x^3).$$

Therefore, uniformly for $x \in [0, n^{2/3}]$

$$\log(1 + x/n) + \log(1 - x/n) = -(x/n)^2 + O(n^2/n^3).$$

Then we have for $n \to \infty$,

$$\int_0^{n^{2/3}} (\log(1 + x/n) + \log(1 - x/n)) \; dx = \int_0^{n^{2/3}} -(x/n)^2 dx + O(n^{-1/3}) = -1/3 + O(n^{-1/3}),$$

and hence

$$\lim_{n\to\infty} \int_0^{n^{2/3}} (\log(1 + x/n) + \log(1 - x/n)) \; dx = -1/3.$$

**Problem 5.** Let

$$f_n(x) = (\log(1 + x/n) + \log(1 - x/n)) \, I_{[0,n/2]}(x).$$

a. Find the function $g(x)$ such that $\lim_{n\to\infty} f_n(x) = g(x)$ for all $x \in \mathbb{R}$.

b. Determine if $f_n \to g$ uniformly.

c. Determine if $f_n \to g$ in $L_1$ or in measure.

**Solution:**
It is easy to see that for each fixed $x \in \mathbb{R}$

$$\lim_{n\to\infty} (\log(1 + x/n) + \log(1 - x/n))\, I_{[0,n/2]}(x) = 0.$$

Therefore, $g(x) = 0$. Clearly,

$$\sup_{x\in\mathbb{R}} |f_n(x) - g(x)| \geq |f_n(n/2)| = \log(4/3),$$

so $f_n \to g$ not uniformly. Also for $x \in [n/4, n/2]$

$$|f_n(x)| = |\log(1 - x^2/n^2)| \geq |f_n(n/4)| = \log(16/15).$$

Therefore, $f_n$ is not convergent in measure, and hence not in $L_1$.

**Problem 6.**
Let $(\mathbb{R}, \mathcal{B})$ be a measurable space, where $\mathcal{B}$ is the Borel $\sigma-$algebra.
a. Prove that $m : \mathcal{B} \to [0, +\infty)$ that given by

$$m(A) := \int_A e^{-x^2} dx$$

is a measure on $\mathcal{B}$. Find $m(C)$, where $C$ is the Cantor set.
b. Compute

$$\int_0^\infty x dm.$$

Try to find the answer in a nice form.
**Solution:**
The fact that $m$ is a measure easy to check by Tonelli's theorem. Also, $m(C) := \int_C e^{-x^2} dx = 0$, since Lebesgue measure of $C$ is 0. Finally, by change of variables $u = x^2$

$$\int_0^\infty x dm = \int_0^\infty x e^{-x^2} dx = 1/2.$$