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Problem 1. Let

$$A = \bigcap_{n=1}^{\infty} \left(\bigcup_{m=1}^{\infty} \left[\frac{1}{mn+1}, \infty \right) \right).$$

Find $m(A)$, where m denotes the Lebesgue measure on $\mathbb{R}$.

Problem 2.
Does exist an open set $A \subset (0,1)$ with the following two properties:
1. $m(A) > 0$;
2. A doesn't contain any binary rational?

Problem 3.
Let

$$f(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} I_{[m^2,\, m^2+3^{-(m+n)}]}(x).$$

a. Prove that f is Lebesgue measurable.
b. Find the Lebesgue integral

$$\int_{\mathbb{R}} f(x)dx.$$

Problem 4. a. Find the following limit

$$\lim_{n\to\infty} \int_1^{\infty} (\log x)^n e^{-x} dx.$$

b. Prove that the function

$$f(t) = \int_1^{\infty} e^{-x^3} \cos xt \, dx$$

is continuous in each point $t \in \mathbb{R}$.
c. Find the following limit

$$\lim_{n\to\infty} \int_0^{n^{2/3}} \left(\log(1+x/n) + \log(1-x/n)\right) \, dx.$$

Hint: First prove that corresponding functions are measurable.

Problem 5. Let

$$f_n(x) = \left(\log(1+x/n) + \log(1-x/n)\right) I_{[0,n/2]}(x).$$

a. Find the function $g(x)$ such that $\lim_{n\to\infty} f_n(x) = g(x)$ for all $x \in \mathbb{R}$.

b. Determine if $f_n \to g$ uniformly.

c. Determine if $f_n \to g$ in L_1 or in measure.

Problem 6.
Let $(\mathbb{R}, \mathcal{B})$ be a measurable space, where $\mathcal{B}$ is the Borel $\sigma-$algebra.
a. Prove that $m : \mathcal{B} \to [0, +\infty)$ that given by

$$m(A) := \int_A e^{-x^2} dx$$

is a measure on $\mathcal{B}$. Find $m(C)$, where C is the Cantor set.
b. Compute

$$\int_0^\infty xdm.$$

Try to find the answer in a nice form.