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Problem 1. Let

$$A = \bigcup_{x>0} [-1, \frac{4x}{x^2+4}).$$

Find $m(A)$, where m denotes the Lebesgue measure on $\mathbb{R}$.
Solution: Since

$$\sup_{x>0} \frac{4x}{x^2+4} = 1,$$

which attains at $x = 2$, we have that $A = [-1, 1)$. Hence, $m(A) = 2$.

Problem 2.
Find a closed set $K \subset (0, 1)$ with $m(K) > 2/3$ such that for any $(a, b) \subset \mathbb{R}$, with $b > a$, $m(K \cap (a, b)) < b - a$. Does exist such a set K with $m(K) = 1$?
Solution: Let $q_1, q_2, \dots$ be an enumeration of all rationals. Take

$$K = [1/20, 19/20] \setminus \bigcup_{n=1}^{\infty} (q_n - \frac{1}{20 \cdot 2^n}, q_n + \frac{1}{20 \cdot 2^n}).$$

It is a closed subset of $(0, 1)$ of measure at most $4/5$, which satisfies $m(K \cap (a, b)) < b - a$ for all $b > a$. There is no such a closed subset of measure 1, since it is a subset of $[\varepsilon, 1 - \varepsilon]$ for some $\varepsilon > 0$.

Problem 3.
Let

$$f(x) = \sum_{n=1}^{\infty} (-1)^{n-1} 2^{-n} I_{[0,n]}(x).$$

a. Prove that f is Lebesgue measurable.
b. Find the Lebesgue integral

$$\int_{\mathbb{R}} f(x) dx.$$

Solution: Clearly,

$$f(x) = \sum_{k=0}^{\infty} c_k I_{(k,k+1)}(x), \text{ a.e.}$$

(exept $x = 0, 1, \dots$), where

$$c_k = \sum_{m=k+1}^{\infty} (-1)^{m-1} 2^{-m} = \frac{(-1)^k 2^{-k}}{3},$$

Therefore, f is measurable. Also, since

$$\sum_{k=0}^{\infty} |c_k| < +\infty,$$

$$\int_{\mathbb{R}} f(x)dx = \sum_{k=0}^{\infty} c_k = \sum_{k=0}^{\infty} \frac{(-1)^k 2^{-k}}{3} = \frac{2}{9}.$$

Problem 4. Find the following limits

a.

$$\lim_{n\to\infty} \int_0^{\infty} x^{3n} e^{-nx} dx.$$

b.

$$\lim_{n\to\infty} \int_{\mathbb{R}} \frac{\sin nx}{x^2+n^2} dx.$$

c^*.

$$\lim_{n\to\infty} \int_0^{n^{2/3}} \left(\frac{1}{2} - \frac{1}{e^{x/n}+e^{-x/n}}\right) dx.$$

Hint: First prove that corresponding functions are measurable for each $n \in \mathbb{N}$.

Solution: Clearly all the functions are measurable since they are products of a continuous function and an indicator function.

a. The function is unsigned. Since $x^3 e^{-x} > 1$ in some neighborhood of $x = 3$ we immediately obtain by Markov's inequality that

$$\lim_{n\to\infty} \int_0^{\infty} x^{3n} e^{-nx} dx = +\infty.$$

b. For all $x \in \mathbb{R}$ and $n \in \mathbb{N}$

$$\left|\frac{\sin nx}{x^2+n^2}\right| \le \frac{1}{x^2+1} \in L_1(\mathbb{R}),$$

therefore by Dominated Convergence Theorem we obtain

$$\lim_{n\to\infty} \int_{\mathbb{R}} \frac{\sin nx}{x^2+n^2} dx = \int_{\mathbb{R}} \lim_{n\to\infty} \frac{\sin nx}{x^2+n^2} dx = 0.$$

c. Since, $e^x = 1 + x + x^2/2 + O(x^3)$, when $x \to 0$, we obtain that

$$\frac{1}{2} - \frac{1}{e^{x/n}+e^{-x/n}} = \frac{x^2}{4n^2} + O(1/n),$$

uniformly in $x \in [0, n^{2/3}]$. So,

$$\lim_{n\to\infty} \int_0^{n^{2/3}} \left(\frac{1}{2} - \frac{1}{e^{x/n}+e^{-x/n}}\right) dx = \lim_{n\to\infty} \int_0^{n^{2/3}} \frac{x^2}{4n^2} dx = \frac{1}{12}.$$

Problem 5. Let

$$f_n(x) = \frac{1}{e^{x/n}+e^{-x/n}} I_{[0,n^{2/3})}(x).$$

a. Find the function $g(x)$ such that $\lim_{n\to\infty} f_n(x) = g(x)$ for all $x \in \mathbb{R}$.

b. Determine if $f_n \to g$ uniformly.

c. Determine if $f_n \to g$ in L_1 or in measure.

Solution: a. For each $x \in \mathbb{R}$,

$$g(x) = \frac{1}{2} I_{[0,\infty)}(x)$$

b. Clearly for each $n \in \mathbb{N}$,

$$\sup_{x\in\mathbb{R}} |f_n(x) - g(x)| \geq |f_n(2n) - g(2n)| = 1/2,$$

therefore the convergence is not uniform.
c. It is easy to show that

$$m(\{x : |f_n(x) - g(x)| \geq 1/4\}) = +\infty,$$

so $f_n \not\to g$ in L_1 or in measure.

Problem 6.
Let $(\mathbb{R}, 2^{\mathbb{R}})$ be a measurable space.

a. Prove that $m(A) = |A \cap \mathbb{N}|$ is a measure on the $\sigma-$algebra $2^{\mathbb{R}}$.
b. Prove that the function

$$f(x) = \frac{\sin(\pi x/2)}{x^2} - \frac{\cos(\pi x/2)}{(x+1)^2}$$

is measurable and compute

$$\int_{\mathbb{R}} f dm.$$

Try to find the answer in a nice form.

Solution: a. Clearly, $m(\emptyset) = 0$, and also for disjoint $A_1, A_2, \ldots$

$$|\bigcup_{n=1}^{\infty} A_n \cap \mathbb{N}| = \bigcup_{n=1}^{\infty} |A_n \cap \mathbb{N}|.$$

b. Any function is measurable in this $\sigma-$algebra, moreover

$$\int_{\mathbb{R}} f dm = \sum_{n=1}^{\infty} f(n) = 1,$$

by telescoping.