Norwegian University of Science and Technology
Department of Mathematical Sciences

Problem 1. Let

$$A = \bigcup_{x>0} [-1, \frac{4x}{x^2+4}).$$

Find $m(A)$, where m denotes the Lebesgue measure on $\mathbb{R}$.

Problem 2.
Find a closed set $K \subset (0,1)$ with $m(K) > 2/3$ such that for any $(a,b) \subset \mathbb{R}$, with $b > a$, $m(K \cap (a,b)) < b - a$. Does exist such a set K with $m(K) = 1$?

Problem 3.
Let

$$f(x) = \sum_{n=1}^{\infty} (-1)^{n-1} 2^{-n} I_{[0,n]}(x).$$

a. Prove that f is Lebesgue measurable.
b. Find the Lebesgue integral

$$\int_{\mathbb{R}} f(x) dx.$$

Problem 4. Find the following limits
a.

$$\lim_{n\to\infty} \int_0^{\infty} x^{3n} e^{-nx} dx.$$

b.

$$\lim_{n\to\infty} \int_{\mathbb{R}} \frac{\sin nx}{x^2+n^2} dx.$$

c^*.

$$\lim_{n\to\infty} \int_0^{n^{2/3}} \left(\frac{1}{2} - \frac{1}{e^{x/n} + e^{-x/n}} \right) dx.$$

Hint: First prove that corresponding functions are measurable for each $n \in \mathbb{N}$.

Problem 5. Let

$$f_n(x) = \frac{1}{e^{x/n} + e^{-x/n}} I_{[0,n^{2/3})}(x).$$

a. Find the function $g(x)$ such that $\lim_{n\to\infty} f_n(x) = g(x)$ for all $x \in \mathbb{R}$.

b. Determine if $f_n \to g$ uniformly.

c. Determine if $f_n \to g$ in L_1 or in measure.

Problem 6.
Let $(\mathbb{R}, 2^{\mathbb{R}})$ be a measurable space.
a. Prove that $m(A) = |A \cap \mathbb{N}|$ is a measure on the $\sigma-$algebra $2^{\mathbb{R}}$.
b. Prove that the function

$$f(x) = \frac{\sin(\pi x/2)}{x^2} - \frac{\cos(\pi x/2)}{(x+1)^2}$$

is measurable and compute

$$\int_{\mathbb{R}} f dm.$$

Try to find the answer in a nice form.