NTNU
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Department of Mathematical Sciences

Examination paper for **TMA4225 Foundations of Analysis**

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Examination date: 21 December, 2021

Examination time (from–to): 09:00–13:00

Permitted examination support material: You need nothing but a pen/pencil, your head and a good mood!

Other information:
The exam contains 12 questions. Each solution will be graded as *rudimentary* (F), *acceptable* (D), *good* (C) or *excellent* (A). Five acceptable solutions guarantee an E; seven acceptable with at least one good a D; seven acceptable with at least five good a C; nine good with at least two excellent a B; nine good with at least seven excellent an A. These are guaranteed limits. Beyond that, the grade is based on the total achievement. Good luck!

Language: English

Number of pages: 4

Number of pages enclosed: 0

Checked by:

Informasjon om trykking av eksamensoppgave	
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Date Signature

Problem 1. Find

$$m(\bigcap_{n=1}^{\infty}[\log n, \infty)),$$

where m denotes the Lebesgue measure on $\mathbb{R}$.
Solution: Since

$$\bigcap_{n=1}^{\infty}[\log n, \infty) = \emptyset$$

then is has measure 0.

Problem 2. Find a Borel measurable set $E \subset \mathbb{R}$ that satisfies simultaneously the following conditions:
1. E is uncountable,
2. E has Lebesgue measure 0,
3. E is dense in $\mathbb{R}$.

Solution: One can take $E = C \cup Q$, where C is a Cantor set.

Problem 3^***.**
For a set $A \subset \mathbb{R}$ denote

$$A + A := \{x + y | \, x, y \in A\}.$$

For example, if $A = \{1, 2\}$, then $A + A = \{2, 3, 4\}$. Find a set A of Lebesgue measure 0 such that $A + A = \mathbb{R}$.
Hint: One can take $A = C + \mathbb{Z}$ and then use the fact that $[0, 1] \subset C + C$.

Problem 4.
Let

$$f(x) = \sum_{n=1}^{\infty} I_{[0, \frac{1}{n(n+1)}]}(x).$$

a. Prove that f is unsigned measurable.
b. Find the Lebesgue integral

$$\int_{\mathbb{R}} f(x)dx.$$

c. Find the smallest $p > 1$ such that $f \notin L_p(\mathbb{R})$. Give the arguments.

Solution:a. f is unsigned measurable because it is pointwise limit of a sequence of unsigned simple functions

$$f(x) = \lim_{k\to\infty} \sum_{n=1}^{k} I_{[0,\frac{1}{n(n+1)}]}(x).$$

b. By Tonelli's theorem

$$\int_{\mathbb{R}} f(x)dx = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{M\to\infty} \sum_{n=1}^{M} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \lim_{M\to\infty} 1 - 1/(M+1) = 1.$$

c. Answer $p = 2$. Clearly,

$$f(x) = \begin{cases} 0, & x < 0, \\ \infty, & x = 0, \\ n, & x \in (\frac{1}{(n+1)(n+2)}, \frac{1}{n(n+1)}],\ n \in \mathbb{N}, \\ 0, & x > 1/2. \end{cases}$$

Therefore,

$$\int_{\mathbb{R}} |f(x)|^p dx = \sum_{n=1}^{\infty} \frac{2n^p}{n(n+1)(n+2)}.$$

The last series is convergent for $p < 2$ and divergent for $p \geq 2$ because

$$\sum_{n=1}^{\infty} \frac{1}{n^q}$$

is convergent only for $q > 1$.

Problem 5. Find the following limits

a.

$$\lim_{n\to\infty} \int_0^{\infty} nxe^{-nx} dx.$$

b.

$$\lim_{t\to 0} \int_{\mathbb{R}} \frac{\cos xt}{x^2 - 2x + 2} dx.$$

c.

$$\lim_{n\to\infty} \int_0^2 x^n (2-x)^n dx.$$

Hint: First prove that corresponding functions are measurable for each $n \in \mathbb{N}$.

Solution: It is easy to check that that all the functions are continuous and therefore measurable.

a. By linear change of variables $y = nx$ we get that for eac $n \in \mathbb{N}$

$$\int_0^\infty nxe^{-nx}dx = \frac{1}{n}\int_0^\infty nxe^{-nx}dnx = \frac{1}{n}\int_0^\infty ye^{-y}dy.$$

It is easy to show now that $ye^{-y} \in L_1[0,\infty]$ since it is continuous bounded and also

$$\lim_{y\to\infty}\frac{ye^{-y}}{1/y^2} = 0.$$

Therefore

$$\lim_{n\to\infty}\int_0^\infty nxe^{-nx}dx = 0.$$

b. Take any sequence t_n such that $\lim_{n\to\infty} t_n = 0$ and denote

$$f_n(x) = \frac{\cos xt_n}{x^2 - 2x + 2}.$$

Since $|f_n(x)| \le h(x)$ for all $x \in \mathbb{R}$, where

$$h(x) = \frac{1}{x^2 - 2x + 2},$$

and $h \in L_1(\mathbb{R})$, we get by Dominated Convergence Theorem that

$$\lim_{n\to\infty}\int_{\mathbb{R}}\frac{\cos xt_n}{x^2 - 2x + 2}dx = \int_{\mathbb{R}}\lim_{n\to\infty}\frac{\cos xt_n}{x^2 - 2x + 2}dx = \int_{\mathbb{R}}\frac{1}{x^2 - 2x + 2}dx = \pi.$$

The last identity can be easily obtained using Riemann integral. Since t_n was arbitrary this implies that

$$\lim_{t\to 0}\int_{\mathbb{R}}\frac{\cos xt}{x^2 - 2x + 2}dx = \pi.$$

c. Denote $f_n(x) := x^n(2-x)^n I_{[0,2]}$. First it is clear that $0 \le x(2-x) \le 1$ for all $x \in [0,2]$ and therefore $|f_n(x)| \le I_{[0,2]}(x)$ for $x \in \mathbb{R}$. Also,

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0, & x \ne 1, \\ 1, & x = 1. \end{cases}$$

Finally by DCT

$$\lim_{n\to\infty}\int_0^2 x^n(2-x)^n dx = \int_0^2 \lim_{n\to\infty} f_n(x)dx = 0.$$

Problem 6. Let
$$f_n(x) = nxe^{-nx}I_{[0,\infty)}(x).$$

a. Find the function $g(x)$ such that $\lim_{n\to\infty} f_n(x) = g(x)$ for all $x \in \mathbb{R}$.

b. Determine if $f_n \to g$ uniformly or almost uniformly.

c. Determine if $f_n \to g$ in L_1 or in measure.

Solution: a. Clearly $g(x) = 0$ as it is easy to check both for $x > 0$ and $x \le 0$.
b. Since for each $n \in \mathbb{N}$

$$\sup_{x\in\mathbb{R}} |f_n(x) - g(x)| \ge |f_n(1/n) - g(1/n)| = 1/e,$$

$f_n \not\to g$ uniformly. Now we will check that for every $\varepsilon > 0$

$$\sup_{x\in\mathbb{R}\setminus[0,\varepsilon]} |f_n(x) - g(x)| \to 0,$$

as $n \to \infty$, which will imply almost uniform convergence. Clearly,

$$\sup_{x\in\mathbb{R}\setminus[0,\varepsilon]} |f_n(x) - g(x)| = \sup_{x\in[\varepsilon,\infty]} xne^{-nx} = \sup_{y\in[\varepsilon n,\infty]} ye^{-y} = \varepsilon ne^{-\varepsilon n}$$

for $n > 1/\varepsilon$. Here we use the fact that function ye^{-y} is decreasing on $[1,\infty]$, which is easy to check by taking derivative.
c. Since for each $n \in \mathbb{N}$

$$\lim_{n\to\infty} \int_{\mathbb{R}} |f_n(x) - g(x)| dx = \lim_{n\to\infty} \int_0^\infty nxe^{-nx} dx = 0,$$

by Problem 5.a. Hence $f_n \to 0$ in L_1 and therefore in measure.