NTNU
Norwegian University of
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Department of Mathematical Sciences

Examination paper for **TMA4225 Foundations of Analysis**

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Examination date: 21 December, 2021

Examination time (from–to): 09:00–13:00

Permitted examination support material: You need nothing but a pen/pencil, your head and a good mood!

Other information:
The exam contains 12 questions. Each solution will be graded as *rudimentary* (F), *acceptable* (D), *good* (C) or *excellent* (A). Five acceptable solutions guarantee an E; seven acceptable with at least one good a D; seven acceptable with at least five good a C; nine good with at least two excellent a B; nine good with at least seven excellent an A. These are guaranteed limits. Beyond that, the grade is based on the total achievement. Good luck!

Language: English

Number of pages: 2

Number of pages enclosed: 0

Checked by:

Date Signature

Informasjon om trykking av eksamensoppgave	
Originalen er:	
1-sidig ☐	**2-sidig** ☒
sort/hvit ☒	**farger** ☐
skal ha flervalgskjema ☐	

Problem 1. Find

$$m(\bigcap_{n=1}^{\infty}[\log n, \infty)),$$

where m denotes the Lebesgue measure on $\mathbb{R}$.

Problem 2. Find a Borel measurable set $E \subset \mathbb{R}$ that satisfies simultaneously the following conditions:
1. E is uncountable,
2. E has Lebesgue measure 0,
3. E is dense in $\mathbb{R}$.

Problem 3*.
For a set $A \subset \mathbb{R}$ denote

$$A + A := \{x + y |\, x, y \in A\}.$$

For example, if $A = \{1, 2\}$, then $A + A = \{2, 3, 4\}$. Find a set A of Lebesgue measure 0 such that $A + A = \mathbb{R}$.

Problem 4.
Let

$$f(x) = \sum_{n=1}^{\infty} I_{[0,\frac{1}{n(n+1)}]}(x).$$

a. Prove that f is unsigned measurable.
b. Find the Lebesgue integral

$$\int_{\mathbb{R}} f(x)dx.$$

c. Find the smallest $p > 1$ such that $f \notin L_p(\mathbb{R})$. Give the arguments.

Problem 5. Find the following limits
a.

$$\lim_{n\to\infty} \int_0^{\infty} nxe^{-nx}dx.$$

b.

$$\lim_{t\to 0} \int_{\mathbb{R}} \frac{\cos xt}{x^2 - 2x + 2}dx.$$

c.

$$\lim_{n\to\infty} \int_0^2 x^n(2-x)^n dx.$$

Hint: First prove that corresponding functions are measurable for each $n \in \mathbb{N}$.

Problem 6. Let

$$f_n(x) = nxe^{-nx} I_{[0,\infty)}(x).$$

a. Find the function $g(x)$ such that $\lim_{n\to\infty} f_n(x) = g(x)$ for all $x \in \mathbb{R}$.

b. Determine if $f_n \to g$ uniformly or almost uniformly.

c. Determine if $f_n \to g$ in L_1 or in measure.