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SOLUTIONS TO THE  
 EXAMINATION IN TMA 4225  
 ANALYSSENS GRUNNLAG  
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PROBLEM 1

Let  $f(x) = \sum_{m=1}^{\infty} (-1)^m \frac{x}{n^3} X_{(n, n+1]}$

(a) Show that  $f$  is Lebesgue integrable.

First we observe that

$$f(x) = \lim_{N \rightarrow \infty} \sum_{m=1}^N (-1)^m \frac{x}{n^3} X_{(n, n+1]}$$

$(-1)^m \frac{x}{n^3}$  is measurable and  $X_{(m, m+1)}$  is measurable so  $(-1)^m \frac{x}{n^3} X_{(m, m+1)}$  is measurable  $\Rightarrow$

$$\sum_{m=1}^N (-1)^m \frac{x}{n^3} X_{(m, m+1)} \text{ is measurable}$$

so  $f$  is the pointwise limit of a measurable function, hence  $f$  is measurable.

(2)

$$|g(x)| \leq \sum_{n=1}^{\infty} |(-1)^n \frac{x}{n^3}| X_{(n, n+1]}$$

$$= \sum_{n=1}^{\infty} \frac{|x|}{n^3} X_{(n, n+1)}$$

Let  $g_{s_2} = \sum_{n=1}^{s_2} \frac{|x|}{n^3} X_{(n, n+1)}$

then  $g_{s_2}(x) \leq 2$  for all  $x$ .

$$\int_R^{\infty} |g(x)| dx = \lim_{s_2 \rightarrow \infty} \int_R^{s_2} g_{s_2}(x) dx$$

$$= \lim_{s_2 \rightarrow \infty} \sum_{n=1}^{s_2} \int_R^{s_2} \frac{|x|}{n^3} X_{(n, n+1)} dx$$

$$= \lim_{s_2 \rightarrow \infty} \sum_{n=1}^{s_2} \int_n^{n+1} \frac{|x|}{n^3} dx$$

(Bounded convergence theorem.)

$$\begin{aligned} &= \lim_{s_2 \rightarrow \infty} \sum_{m=1}^{s_2} \frac{1}{2} \frac{((m+1)^2 - m^2)}{m^3} = \lim_{s_2 \rightarrow \infty} \sum_{m=1}^{s_2} \frac{1}{2} \frac{2m+1}{m^3} \\ &\leq \lim_{s_2 \rightarrow \infty} \sum_{m=1}^{s_2} \frac{2}{m^2} < \infty. \end{aligned}$$

(3)

e)

Find a formula for

$$\int_{\mathbb{R}} f(x) dx, \quad f_n(x) = \sum_{m=1}^n (-1)^m \frac{x}{m^3} \chi_{(m, m+1]}$$

$$\text{Now } |f_n(x)| \leq \sum_{m=1}^{\infty} \frac{|x|}{m^3} \chi_{(m, m+1]} = g(x)$$

$$\text{and } \int_{\mathbb{R}} g(x) dx < 0 \rightarrow 0$$

the dominate convergence theorem implies that

$$\begin{aligned} & \int_{\mathbb{R}} \left( \sum_{m=1}^{\infty} (-1)^m \frac{x}{m^3} \chi_{(m, m+1]} \right) \\ &= \sum_{m=1}^{\infty} \int_{\mathbb{R}} (-1)^m \frac{x}{m^3} \chi_{(m, m+1]} \\ &= \sum_{m=1}^{\infty} (-1)^m \int_{(m, m+1]} \frac{x}{m^3} dx = \sum_{m=1}^{\infty} (-1)^m \frac{1}{2} \frac{(m+1)^2 - m^2}{m^3} \\ &= \sum_{m=1}^{\infty} (-1)^m \frac{1}{2} \frac{2m+1}{m^3} \end{aligned}$$

(4)

Problem 2:

$(X, \mathcal{S}, \mu)$  is a measure space  
 and  $f \in L^n(\mu)$ ,  $1 < n < \infty$

This means that  $\left( \int_X |f|^n d\mu \right)^{1/n} < \infty$ ,

hence  $\int_X |f|^n d\mu < \infty$  or

$|f|^n \in L^1(\mu)$  so it follows

inequality implies that

$$\mu(\{x \in X : |f(x)|^n > c\}) \leq \frac{1}{c} \|f^n\|_1$$

Now

$$\{x \in X : |f(x)| \geq b\} =$$

$$\{x \in X : |f(x)|^n \geq b^n\}.$$

So

$$\mu(\{x \in X : |f(x)| \geq b\}) =$$

$$\mu(\{x \in X : |f(x)|^n \geq b^n\}) \leq \frac{1}{b^n} \int_X |f(x)|^n d\mu$$

Hence

$$\lambda_2^{n-1} \{x \in X : |g(x)| \geq \lambda_2\}$$

$$\leq \frac{\lambda_2^{n-1}}{\lambda_2^n} \int_X |g(x)|^n d\mu = \frac{1}{\lambda_2} \int_X |g(x)|^n d\mu$$

$\rightarrow 0$

$$\lim_{\lambda_2 \rightarrow \infty} \lambda_2^{n-1} \mu(\{x \in X : |g(x)| \geq \lambda_2\}) = 0$$

Since  $\int_X |g(x)|^n d\mu < \infty$

### PROBLEM 3:

Assume  $g : [0, 1] \rightarrow \mathbb{R}$  is in  $L^1([0, 1])$ , then we can find a bounded  $f$  on  $[0, 1]$  such that

$$\int_{[0, 1]} f g d\lambda = \|g\|_1, \|f\|_\infty$$

Let  $f(x) = \begin{cases} 1 & \text{if } g(x) \geq 0 \\ -1 & \text{if } g(x) < 0 \end{cases}$

Then  $\|f\|_\infty = 1$

G

and

$$f(x)g(x) = |g(x)| \rightarrow 0$$

$$\int_{[0,1]} f(x)g(x) d\lambda = \int_{[0,1]} |g(x)| d\lambda = \|g\|_1$$

$$= \|g\|_1 \cdot 1 = \|g\|_1 \|f\|_\infty.$$

### Problem 4

Let  $(X, \mathcal{S}, \mu)$  be a measure space and assume that  $f: X \rightarrow [0, \infty]$  is measurable and integrable.

$$\omega(A) = \int_X f(x) X_A(x) d\mu$$

(a) Prove that  $\omega$  is a measure on  $(X, \mathcal{S})$

$$\textcircled{i} \quad \omega(\emptyset) = \int_X f(x) X_{\emptyset}(x) d\mu = \int_X 0 d\mu = 0$$

\textcircled{ii} Let  $\{A_j\}$  be a disjoint sequence of sets in  $\mathcal{S}$

$$\text{Then } X_{\bigcup A_j} = \sum_j X_{A_j}$$

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$$\text{So } \omega(\cup A_j) = \int_X f(x) X_{\cup A_j}(x) d\mu$$

$$= \int_X f(x) \sum_{j \in J} X_{A_j}(x) d\mu =$$

( Dominated convergence theorem )

$$\text{since } \sum_j X_{A_j}(x) \leq X_X(x)$$

$$\sum_j \int_X f(x) X_{A_j}(x) d\mu$$

$$= \sum_j \omega(A_j)$$

(b) Prove that if  $g \in L^1(\omega)$ , then

$$\int_X g d\omega = \int_X g g d\mu$$

We shall break the proof into several pieces.

- ①  $g$  is a characteristic function
- ②  $g$  is a simple function.
- ③  $g$  is a non negative function
- ④  $g$  is a  $L^1(\omega)$  function.

① If  $g = X_A$  for some  $A \in S$ ,

$$\text{then } \int g d\omega = \int_{\Omega} X_A d\omega$$

$$= \omega(A) = \int g X_A d\mu$$

②

~~given~~

$$g = c_1 X_{A_1} + \dots + c_n X_{A_n}$$

where  $\{A_i\}$  are disjoint,

$$\text{then } \int g d\omega = \int_{\Omega} c_1 X_{A_1} d\omega + \dots + c_n \int_{\Omega} X_{A_n} d\omega$$

$$\begin{aligned}
 \text{so } \int_X g d\nu &= c_1 \nu(A_1) + \dots + c_n \nu(A_n) \\
 &= c_1 \int_X \chi_{A_1} g d\mu + \dots + c_n \int_X \chi_{A_n} g d\mu \\
 &= \int_X (c_1 \chi_{A_1} g + \dots + c_n \chi_{A_n} g) d\mu \\
 &= \int_X g f d\mu.
 \end{aligned}$$

(3) If  $g$  is non-negative, then  
 it is an increasing limit  
 of simple functions  $g_k$ , then  
 the monotone limit theorem implies  
 that

$$\int g d\nu = \lim_{k \rightarrow \infty} \int g_k d\nu = \lim_{k \rightarrow \infty} \int g_k f d\mu$$

(Now  $g_k f$  increases up  
 towards  $gf$ )

$$= \int (\lim g_k f) d\mu = \int g f d\mu$$

Problem 5:

Assume that  $f : (0, \infty) \rightarrow \mathbb{R}$  and  $f \in L^p(\lambda)$  for some  $1 < p < \infty$ , then

$$\left| \int_0^x f(t) dt \right| \leq \|f\|_p \cdot x^{1 - \frac{1}{p}}$$

for all  $x > 0$ .

Let  $\alpha'$  be such that

$$\frac{1}{p} + \frac{1}{\alpha'} = 1 \quad \text{and for}$$

a fixed  $x > 0$  let  $g(t) = X_{(0,x)}^{(t)}$

Then  $g \in L^{\alpha'}(\lambda)$

Now the Hölder inequality implies that

$$\int_{(0,\infty)} |fg| d\lambda \leq \|f\|_p \|g\|_{\alpha'}$$

$$\text{Now } \|g\|_{\alpha^1} = \left( \int_{(0,\infty)} |g|^{\alpha^1} d\lambda \right)^{\frac{1}{\alpha^1}}$$

$$= \left( \int_{(0,x)} X_{(0,x)}^{\alpha^1} d\lambda \right)^{\frac{1}{\alpha^1}}$$

$$= \left( \int_{(0,x)} X_{(0,x)} d\lambda \right)^{\frac{1}{\alpha^1}}$$

$$= \left( \int_0^x 1 d\lambda \right)^{\frac{1}{\alpha^1}} = (x)^{\frac{1}{\alpha^1}} = x^{1-\frac{1}{\alpha^1}}$$

### Problem 6:

$f$  is Lebesgue measurable and bounded on  $[0,1]$

Assume  $\int_{[0,1]} |f(x)| d\lambda > 0$ , this is

just to make sure we do divide by 0 in the limit

below.

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$$\text{Then } \lim_{n \rightarrow \infty} \frac{\int_{[0,1]} |g|^{m+1} d\lambda(x)}{\int_{[0,1]} |g|^m d\lambda(x)} = \|g\|_\infty$$

Hence

Let  $0 < \alpha < 1$  and

$$E = \{x \in [0,1] : \alpha \|g\|_\infty < |g(x)|\}$$

$$\text{Then } \lim_{m \rightarrow \infty} \frac{\int_{[0,1] \setminus E} |g|^m d\lambda}{\int_{[0,1]} |g|^m d\lambda} = 0$$

We may assume that  $\|g\|_\infty = 1$

otherwise we divide  $g$  by

$\|g\|_\infty$  which is assumed to

be larger than 0.

On  $[0,1] \setminus E$  we know

that  $|g(x)| \leq \alpha < 1$ .

Let  $1 > \beta > \alpha$ , then we know  
 that  $\lambda\{x : |g(x)| > \beta\} = a > 0$

Now

$$\frac{\int_{[0,1] \setminus E} |g|^n dx}{\int_{[0,1]} |g|^n dx} \leq \frac{\int_{[0,1] \setminus E} |g|^n dx}{\int_{\{x : |g(x)| > \beta\}} |g|^n dx}$$

$$\leq \frac{\alpha^n \lambda([0,1] \setminus E)}{\beta^n \lambda(\{x : |g(x)| > \beta\})}$$

$$\leq \frac{\alpha^n}{\beta^n} \frac{1}{a}$$

Since  $0 < \frac{\alpha}{\beta} < 1$  it follows

that  $\frac{\alpha^n}{\beta^n} \rightarrow 0$  as  $n \rightarrow \infty$

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First observe that

$$\int_{[0,1]} |f|^{n+1} d\lambda = \int_{[0,1]} |f|^n |f| d\lambda$$

use Hölder with

$$p=1 \quad \text{and} \quad p'=\infty$$

it follows that

$$\int_{[0,1]} |f|^{n+1} d\lambda \leq \left( \int_{[0,1]} |f|^n d\lambda \right) \|f\|_\infty$$

Hence

$$\frac{\int_{[0,1]} |f|^{n+1} d\lambda}{\int_{[0,1]} |f|^n d\lambda} \leq \|f\|_\infty$$

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Given any  $\varepsilon > 0$  and  $\alpha < 1$   
 with a large  $m$  we  
 obtain :

$$\frac{\int_{[0,1]} |f|^{m+1} d\lambda}{\int_{[0,1]} |f|^m d\lambda} = \frac{\int_E |f|^{m+1} d\lambda + \int_{[0,1] \setminus E} |f|^{m+1} d\lambda}{\int_E |f|^m d\lambda + \int_{[0,1] \setminus E} |f|^m d\lambda}$$

$$= \frac{\int_E |f|^{m+1} d\lambda}{\int_E |f|^m d\lambda} \left( 1 + \frac{\int_{[0,1] \setminus E} |f|^{m+1} d\lambda}{\int_E |f|^{m+1} d\lambda} \right) \\ \quad \left( 1 + \frac{\int_{[0,1] \setminus E} |f|^m d\lambda}{\int_E |f|^m d\lambda} \right)$$

$$\geq \frac{\int_E |f|^{m+1} d\lambda}{\int_E |f|^m d\lambda} \left( \frac{1 - \varepsilon}{1 + \varepsilon} \right)$$

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$$\geq \frac{\alpha \|f\|_{\infty} \int_E |f|^n d\lambda}{\int_E |f|^m d\lambda} \left( \frac{1-\varepsilon}{1+\varepsilon} \right)$$

$$= \|f\|_{\infty} \alpha \left( \frac{1-\varepsilon}{1+\varepsilon} \right)$$

Since  $\alpha$  can be very near 1 and  $\varepsilon$  near 0.

It follows that given any  $\delta > 0$  then

$$\lim_{\delta \rightarrow 0} \frac{\int_{[0,1]} |f|^{n+1} d\lambda}{\int_{[0,1]} |f|^n d\lambda} \geq (1-\delta) \|f\|_{\infty}$$

Problem 7:

a) Use Fubini to show that

$$I = \int_{-\infty}^{\infty} e^{-ex^2} dx = \sqrt{\frac{\pi}{e}}$$

$$I^2 = \left( \int_{-\infty}^{\infty} e^{-ex^2} dx \right) \left( \int_{-\infty}^{\infty} e^{-ey^2} dy \right)$$

$$= \left( \int_{-\infty}^{\infty} e^{-ex^2} dx \right) \left( \int_{-\infty}^{\infty} e^{-ey^2} dy \right)$$

$$\text{Fubini} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-ex^2 - ey^2} dx dy$$

$$= \int_{\mathbb{R}^2} e^{-e(x^2 + y^2)} dx dy$$

Change to polar  
coordinates.

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$$I^2 = \int_0^{2\pi} \int_0^\infty e^{-b r^2} r dr d\theta$$

$$= \int_0^{2\pi} \left( -\frac{1}{2b} e^{-b r^2} \right) \Big|_0^\infty d\theta$$

$$= \int_0^{2\pi} \frac{1}{2b} d\theta = \frac{\pi}{b}$$

$$\text{So } I^2 = \frac{\pi}{b}, \text{ hence}$$

$$I = \sqrt{\frac{\pi}{b}}$$

(e)

$$\int_{\mathbb{R}^n} e^{-(a_1 x_1^2 + \dots + a_n x_n^2)} d\lambda_n(x)$$

$a_1, \dots, a_n > 0$

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We use Fubini

$$\int_{\mathbb{R}^n} e^{-(a_1 x_1^2 - \dots - a_n x_n^2)} d\lambda_n(x)$$

$$= \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} e^{-a_1 x_1^2} e^{-a_2 x_2^2} \dots e^{-a_n x_n^2} dx_1 \dots dx_n$$

$$= \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \sqrt{\frac{\pi}{a_1}} e^{-a_2 x_2^2} \dots e^{-a_n x_n^2} dx_2 \dots dx_n$$

$$= \dots = \sqrt{\frac{\pi}{a_1}} \sqrt{\frac{\pi}{a_2}} \dots \sqrt{\frac{\pi}{a_n}}$$

$$= \sqrt{\frac{\pi^n}{a_1 \dots a_n}}$$