EXAMINATION in TMA 4225

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There are 7 Problems.

**Problem 1**
Let $f : \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) = \sum_{n=1}^{\infty} (-1)^n \frac{x}{n^3} \chi_{(n,n+1]}$$

(a) Show that $f$ is integrable

(b) Compute $\int_{\mathbb{R}} f(x) d\lambda(x)$

**Problem 2**
Assume that $(X, \mathcal{S}, \mu)$ is a measure space and that $f \in L^p(\mu)$ for some $1 < p < \infty$.
Show that $\lim_{k\to\infty} k^{p-1} \mu(\{x \in X; |f(x)| \geq k\}) = 0$.

**Problem 3**
Assume that $g : [0, 1] \to \mathbb{R}$ is integrable. Show that there exists a bounded $f$ on $[0, 1]$ such that

$$\int_{[0,1]} g f d\lambda = \|g\|_1 \|f\|_\infty.$$

**Problem 4**
Let $(X, \mathcal{S}, \mu)$ be a measure space and $f : X \to [0, \infty]$ is integrable. Define the measure $\nu(A) = \int_X f \chi_A d\mu$ for every $A \in \mathcal{S}$.
(a) Show that $\nu$ is a measure on $(X; \mathcal{S})$

(b) Prove that if $g \in L^1(\nu)$, then $\int_X g d\nu = \int_X g f d\mu$.

(Give all the details)

**Problem 5**

Assume that $f : (0, \infty) \to \mathbb{R}$ and that $f \in L^p(\lambda)$ for some $1 < p < \infty$. Show that

$$| \int_0^x f(t)dt| \leq \|f\|_p x^{1-\frac{1}{p}}$$

for all $x > 0$.

**Problem 6**

Assume that $f$ is Lebesgue measureable and bounded on $[0, 1]$.
Prove that if $\int_{[0,1]} |f(x)|d\lambda(x) > 0$, then

$$\lim_{n\to\infty} \frac{\int_{[0,1]} |f(x)|^{n+1} d\lambda(x)}{\int_{[0,1]} |f(x)|^n d\lambda(x)} = \|f\|_\infty.$$

Hint: Let $0 < \alpha < 1$ and $E = \{x \in [0, 1]; \alpha\|f\|_\infty < |f(x)|\}$. Prove that

$$\lim_{n\to\infty} \frac{\int_{[0,1]\setminus E} |f|^n dx}{\int_{[0,1]} |f|^n dx} = 0.$$

**Problem 7**

(a) Use Fubini's Theorem together with polar coordinates to show that

$$\int_{-\infty}^{\infty} e^{-bx^2} dx = \sqrt{\frac{\pi}{b}}$$

when $b > 0$.

(b) Find a formula for $\int_{\mathbb{R}^n} e^{-(a_1 x_1^2 + \cdots + a_n x_n^2)} d\lambda_n(x)$ when $a_1, a_2, \ldots, a_n > 0$.