Problem 1. Find a Lebesgue measurable set $E \subset \mathbb{R}$ such that $E \cap I$ is not Jordan measurable for each nonempty open interval $I \subset \mathbb{R}$.

Solution: Take $E = \mathbb{Q}$. Then $m_*^J(\mathbb{Q} \cap I) = 0$ and $m^{*,J}(\mathbb{Q} \cap I) = |I|$. Therefore $\mathbb{Q} \cap I$ is not Jordan measurable.

Problem 2.

a. Let A be a measurable set such that

$$m(A \cap [x, x+1]) < \frac{1}{(|x|+1)^2}$$

for each $x \in \mathbb{R}$. Prove that $m(A) < +\infty$. Here m denotes the Lebesgue measure.

b. Find a measurable set $A \subset \mathbb{R}$ such that $m(A) = +\infty$ and

$$m(A \cap [x, x+1]) < \frac{1}{(2|x|+1)}$$

for each $x \in \mathbb{R}$.

Solution:

a. Clearly

$$m(A) = \sum_{n=-\infty}^{\infty} m(A \cap [n, n+1]) < \sum_{n=-\infty}^{\infty} \frac{1}{(|n|+1)^2} < \infty.$$

b. Consider

$$A = \cup_{n=1}^{\infty} [n, n + \frac{1}{10n}].$$

Clearly,

$$m(A) = \sum_{n=1}^{\infty} \frac{1}{10n} = +\infty,$$

since all the segments in the union are disjoint. The condition

$$m(A \cap [x, x+1]) < \frac{1}{(2|x|+1)}$$

is easy to check.

Problem 3.

a. Let

$$f(x) = \sum_{q=1}^{\infty} \sum_{p=1}^{q} \frac{1}{q^2(q+1)} e^{-|x-p/q|}.$$

Find all $x \in \mathbb{R}$ such that $f(x) < +\infty$.
b. Prove that f is unsigned measurable.
c. Find the Lebesgue integral

$$\int_{\mathbb{R}} f(x)dx.$$

d. Prove that there are $x, y \in \mathbb{R}$ such that $f(x) \neq f(y)$.

Solution:

a. We have

$$f(x) \leq \sum_{q=1}^{\infty} \sum_{p=1}^{q} \frac{1}{q^2(q+1)} = \sum_{q=1}^{\infty} \frac{1}{q(q+1)} = \sum_{q=1}^{\infty} \left(\frac{1}{q} - \frac{1}{q+1} \right) = 1$$

for all $x \in \mathbb{R}$.
b. For all $p, q \in \mathbb{N}$ the function $e^{-|x-p/q|}$ is continuous and therefore measurable. Hence f is unsigned measurable as countable sum of unsigned measurable functions.
c. By change of variables it is easy to obtain that

$$\int_{\mathbb{R}} e^{-|x-p/q|} dx = \int_{\mathbb{R}} e^{-|x|} dx = 2 \int_0^{\infty} e^{-x} dx = 2.$$

Therefore, by Tonelli's theorem

$$\int_{\mathbb{R}} f(x)dx = \sum_{q=1}^{\infty} \sum_{p=1}^{q} \frac{1}{q^2(q+1)} \int_{\mathbb{R}} e^{-|x-p/q|} dx = \sum_{q=1}^{\infty} \frac{2}{q(q+1)} = 2.$$

d. Assume contrary that for some $a \geq 0$ $f(x) = a$ for all $x \in \mathbb{R}$. Then

$$\int_{\mathbb{R}} f(x)dx = 0 \,\text{or}\, +\infty.$$

This contradicts c.

Problem 4. Find the following limits

a.

$$\lim_{n \to \infty} \int_{\mathbb{R}} \frac{\cos \pi x}{1 + x^n} I_{[0,\infty]}(x) dx.$$

Here $I_B(x)$ stands for the indicator function of a set $B \subset \mathbb{R}$.
b.

$$\lim_{n \to \infty} \int_0^{\infty} \frac{1}{(n-1)!} x^{n^2+n-1} e^{-x^n} dx.$$

Hint: First prove that corresponding functions are measurable for each $n \in \mathbb{N}$.

Solution: a. For each $n \in \mathbb{N}$ denote by

$$f_n(x) := \frac{\cos \pi x}{1 + x^n} I_{[0,\infty]}(x).$$

Each f_n is a product of a function which is continuous on $[0, \infty)$ and $I_{[0,\infty]}(x)$. Therefore all f_n are measurable. We have that

$$f(x) := \lim_{n\to\infty} f_n(x) = \begin{cases} 0, & x < 0, \\ \cos \pi x, & x \in [0, 1), \\ -1/2, & x = 1, \\ 0, & x > 1. \end{cases}$$

Hence,

$$\int_{\mathbb{R}} f(x)dx = \int_0^1 \cos \pi x dx = 0.$$

Also,

$$|f_n(x)| \le g(x), \quad x \in \mathbb{R},$$

for all $n \ge 2$, where

$$g(x) = \begin{cases} 0, & x < 0, \\ 1, & x \in [0, 1], \\ \frac{1}{1+x^2}, & x > 1. \end{cases}$$

Since, $g \in L_1(\mathbb{R})$, then by dominated convergence theorem

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)dx = \int_{\mathbb{R}} f(x)dx = 0.$$

b. All functions are measurable by the same reason as in a. By change of variables $y = x^n$ we have.

$$\int_0^\infty \frac{1}{(n-1)!} x^{n^2+n-1} e^{-x^n} dx = \int_0^\infty \frac{1}{n!} y^n e^{-y} dy = 1.$$

Problem 5. Let

$$f_n(x) = \frac{\cos \pi x}{1 + x^n} I_{[0,\infty)}(x).$$

a. Find the function $g(x)$ such that $\lim_{n\to\infty} f_n(x) = g(x)$ for all $x \in \mathbb{R}$.

b. Determine if $f_n \to g$ uniformly, in L_2, almost uniformly, in L_1, or in measure.

c. Determine if there is $h \in L_1(\mathbb{R})$ such that $|f_n(x)| < h(x)$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$.

Solution: a. By the solution of Problem 4 (a) we see that

$$g(x) = \begin{cases} 0, & x < 0, \\ \cos \pi x, & x \in [0,1), \\ -1/2, & x = 1, \\ 0, & x > 1. \end{cases}$$

b.Clearly

$$\lim_{x\to 1-} |f_n(x) - g(x)| = 1/2$$

for each $n \in \mathbb{N}$. Therefore f_n doesn't converge to g uniformly. On the other hand, for each fixed $\varepsilon > 0$

$$a_n := \sup_{x\in[0,1-\varepsilon]} \left| \frac{\cos \pi x}{1+x^n} - \cos \pi x \right| \le \sup_{x\in[0,1-\varepsilon]} \frac{x^n}{1+x^n} \le \frac{(1-\varepsilon)^n}{2} \to 0, \quad n\to\infty,$$

and

$$b_n := \sup_{x\in[1+\varepsilon,\infty)} \left| \frac{\cos \pi x}{1+x^n} \right| \le \frac{1}{(1+\varepsilon)^n} \to 0, \quad n \to \infty.$$

Therefore

$$\sup_{x\in\mathbb{R}\setminus[1-\varepsilon,1+\varepsilon]} |f_n(x) - g(x)| \to 0, \quad n\to\infty,$$

so $f_n \to g$ almost uniformly. We have that for each fixed $\varepsilon > 0$ and $n \in \mathbb{N}$

$$\int_{\mathbb{R}} |f_n(x) - g(x)| dx = \int_0^{1-\varepsilon} \left| \frac{\cos \pi x}{1+x^n} - \cos \pi x \right| dx + \int_{1-\varepsilon}^{1} \left| \frac{\cos \pi x}{1+x^n} - \cos \pi x \right| dx$$

$$+ \int_1^{\infty} \left| \frac{\cos \pi x}{1+x^n} \right| dx := I_1 + I_2 + I_3.$$

Clearly, by monotonicity of Lebesgue integral $I_1 < (1-\varepsilon)^{n+1}/2$, $I_2 < 2\varepsilon$, $I_3 < \int_1^\infty x^{-n} = 1/(n-1)$. Choosing ε small enough and then sending n to ∞ we get that

$$\int_{\mathbb{R}} |f_n(x) - g(x)| dx \to 0, \quad n \to \infty,$$

therefore $f_n \to g$ in L_1 and hence in measure. Similarly,

$$\int_{\mathbb{R}} |f_n(x) - g(x)|^2 dx \to 0, \quad n \to \infty,$$

so $f_n \to g$ in L_2.

c. There is no such a function h because $f_1 \notin L_1$. It is easy to show that

$$\int_0^\infty \left| \frac{\cos \pi x}{1+x} \right| dx = +\infty.$$