NTNU
Norwegian University of
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Department of Mathematical Sciences

Examination paper for **TMA4225 Foundations of Analysis**

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Examination date: 21 December, 2018

Examination time (from–to): 09:00–13:00

Permitted examination support material: You need nothing but a pen/pencil, your head and a good mood!

Other information:
The exam contains 12 questions. Each solution will be graded as *rudimentary* (F), *acceptable* (D), *good* (C) or *excellent* (A). Five acceptable solutions guarantee an E; seven acceptable with at least one good a D; seven acceptable with at least five good a C; nine good with at least two excellent a B; nine good with at least seven excellent an A. These are guaranteed limits. Beyond that, the grade is based on the total achievement. Good luck!

Language: English

Number of pages: ??

Number of pages enclosed: 0

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Informasjon om trykking av eksamensoppgave	
Originalen er:	
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Date Signature

Problem 1. Find a Lebesgue measurable set $E \subset \mathbb{R}$ such that $E \cap I$ is not Jordan measurable for each non empty open interval $I \subset \mathbb{R}$.

Problem 2.

a. Let A be a measurable set such that

$$m(A \cap [x, x+1]) < \frac{1}{(|x|+1)^2}$$

for each $x \in \mathbb{R}$. Prove that $m(A) < +\infty$. Here m denotes the Lebesgue measure.

b. Find a measurable set $A \subset \mathbb{R}$ such that $m(A) = +\infty$ and

$$m(A \cap [x, x+1]) < \frac{1}{(2|x|+1)}$$

for each $x \in \mathbb{R}$.

Problem 3.

a. Let

$$f(x) = \sum_{q=1}^{\infty} \sum_{p=1}^{q} \frac{1}{q^2(q+1)} e^{-|x-p/q|}.$$

Find all $x \in \mathbb{R}$ such that $f(x) < +\infty$.

b. Prove that f is unsigned measurable.

c. Find the Lebesgue integral

$$\int_{\mathbb{R}} f(x)dx.$$

d. Prove that there are $x, y \in \mathbb{R}$ such that $f(x) \neq f(y)$.

Problem 4. Find the following limits

a.

$$\lim_{n\to\infty} \int_{\mathbb{R}} \frac{\cos \pi x}{1+x^n} I_{[0,\infty]}(x)dx.$$

Here $I_B(x)$ stands for the indicator function of a set $B \subset \mathbb{R}$.

b.

$$\lim_{n\to\infty} \int_0^{\infty} \frac{1}{(n-1)!} x^{n^2+n-1} e^{-x^n} dx.$$

Hint: First prove that corresponding functions are measurable for each $n \in \mathbb{N}$.

Problem 5. Let

$$f_n(x) = \frac{\cos \pi x}{1+x^n} I_{[0,\infty)}(x).$$

a. Find the function $g(x)$ such that $\lim_{n\to\infty} f_n(x) = g(x)$ for all $x \in \mathbb{R}$.

b. Determine if $f_n \to g$ uniformly, in L_2, almost uniformly, in L_1, or in measure.

c. Determine if there is $h \in L_1(\mathbb{R})$ such that $|f_n(x)| < h(x)$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$.