NTNU
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Department of Mathematical Sciences

Examination paper for **TMA4225 Foundations of Analysis**

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Examination time (from–to): 09:00–13:00

Permitted examination support material: You need nothing but a pen/pencil, your head and a good mood!

Other information:
The exam contains 12 questions. Each solution will be graded as *rudimentary* (F), *acceptable* (D), *good* (C) or *excellent* (A). Five acceptable solutions guarantee an E; seven acceptable with at least one good a D; seven acceptable with at least five good a C; nine good with at least two excellent a B; nine good with at least seven excellent an A. These are guaranteed limits. Beyond that, the grade is based on the total achievement. Good luck!

Language: English

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Checked by:

Informasjon om trykking av eksamensoppgave	
Originalen er:	
1-sidig ☐	**2-sidig** ☒
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Date Signature

Problem 1.

a. Prove that there is an open set E in $\mathbb{R}$ such that the following conditions hold simultaneously:
(i) $E \supset Q$, where Q is a set of all rationals;
(ii) $E \supset [0,1]$;
(iii) $m(E) < 2$, where m is a Lebesgue measure.

b. Prove that for each measurable set A in $\mathbb{R}$ the function $f(x) := m(A \cap [0,x])$ is well-defined and continuous on $[0,\infty)$.

c*. Prove that there is an open set E in $\mathbb{R}$ such that the following conditions hold simultaneously:
(i) $E \supset Q$;
(ii) $m(E) = 1$.

Solution: a. Fix $\varepsilon > 0$. Take a bijection $q : \mathbb{N} \to \mathbb{Q}$ and choose

$$E = (-\varepsilon, 1+\varepsilon) \cup \bigcup_{n=1}^{\infty} (q(n) - \frac{\varepsilon}{2^n}, q(n) + \frac{\varepsilon}{2^n}).$$

First, it is clear that E is open since it is a union of open intervals. Then, $E \supset Q$ because q is a bijection. Also, $E \supset [0,1]$. Now, using countable subadditivity and the fact that $m((a,b)) = b - a$ we obtain

$$m(E) \leq 1 + 2\varepsilon + \sum_{n=1}^{\infty} \frac{2\varepsilon}{2^n} \leq 1 + 4\varepsilon < 2$$

for each $\varepsilon < 1/4$. $\square$

b. Note that the set $A \cap [0,x]$ is measurable and also $0 \leq m(A \cap [0,x]) \leq x$ for each $x \geq 0$, therefore f is well-defined. Then it is easy to check that for $0 \leq x < y$ we have that $A \cap [0,x] \subset A \cap [0,y]$ and $(A \cap [0,y]) \setminus (A \cap [0,x]) \subset (x,y]$. Therefore $|f(x) - f(y)| \leq |x - y|$ for all $x, y \geq 0$. This implies continuity of f. $\square$

c. First choose as in a.

$$A := \bigcup_{n=1}^{\infty} (q(n) - \frac{1}{2^{n+2}}, q(n) + \frac{1}{2^{n+2}}).$$

By subadditivity $m(A) \leq 1/2$. Now, similarly to b. it is easy to show that the function $g(x) = m(A \cup (0,x))$ satisfies $|g(x) - g(y)| \leq |x - y|$ for $x, y > 0$ and therefore continuous in each point $x > 0$. Now, by subadditivity $g(1/4) \leq 3/4$ and

$g(2) \geq 2$, so by mean value theorem $g(z) = 1$ for some $z \in (1/4, 2)$. Finally, note that the set $E := A \cup (0, z)$ satisfies all the conditions. $\square$

Problem 2.
a. Prove that the function

$$f(x) = \sum_{n=1}^{\infty} e^{-2^n x} I_{[0,\infty]}(x)$$

is unsigned measurable. Here $I_B(x)$ stands for the indicator function of a set $B \subset \mathbb{R}$.

b. Find the set of all $x \in \mathbb{R}$ for which $f(x) < +\infty$.

c. Find the Lebesgue integral

$$\int_{\mathbb{R}} f(x)dx.$$

Solution: a. Consider a function $g_n(x) = e^{-2^n x}$. It is continuous on $\mathbb{R}$ for all $n \in \mathbb{N}$ and therefore measurable. Also we know that $I_{[0,\infty]}(x)$ is measurable. Now for each $N \in \mathbb{N}$ the function

$$f_N(x) = \sum_{n=1}^{N} e^{-2^n x} I_{[0,\infty]}(x)$$

is measurable since sum and product of two measurable functions is a measurable function. For each $x \geq 0$ we have that $F_1(x) < F_2(x)...$, so $f(x) = sup_{N \in \mathbb{N}} f_N(x)$ is measurable. $\square$

b. Answer: $\mathbb{R} \setminus \{0\}$. Clearly, $f(0) = +\infty$ and $f(x) = 0$ for $x < 0$. Also for each $x > 0$

$$f(x) \leq \sum_{n=1}^{\infty} e^{-nx} = \frac{e^{-x}}{1 - e^{-x}} < +\infty.$$

$\square$

c. By Tonelli's theorem for sums and integrals we have

$$\int_{\mathbb{R}} f(x)dx = \sum_{n=1}^{\infty} \int_{\mathbb{R}} e^{-2^n x} I_{[0,\infty]}(x)dx.$$

Now, by truncation property and corresponding identity between Riemann and Lebesgue integral we have for each $n \in \mathbb{N}$

$$\int_{\mathbb{R}} e^{-2^n x} I_{[0,\infty]}(x)dx = \lim_{M \to \infty} \int_0^M e^{-2^n x}dx = 2^{-n}.$$

Therefore

$$\int_{\mathbb{R}} f(x)dx = \sum_{n=1}^{\infty} 2^{-n} = 1.$$

□

Problem 3. Let $\mathcal{B}$ be a null σ-algebra on $\mathbb{R}$, that is the σ-algebra that consists of the sets of Lebesgue measure 0 and their complements. Prove that $\mathcal{B}$ is not atomic.

Solution: Let A be an atom. Assume that $x, y \in A$, where $x, y \in \mathbb{R}$ and $x \neq y$ then all the sets in the σ-algebra contain both x, y or none of them. This is a contradiction because one point sets $\{x\}$ and $\{y\}$ are in $\mathcal{B}$. So all atoms consists of one point, hence $\mathcal{B} = 2^{\mathbb{R}}$. On the other hand, it is clear that not all the subsets of $\mathbb{R}$ has measure 0. This implies that $\mathcal{B}$ is not atomic. □

Problem 4. Find the following limits

a.

$$\lim_{n\to\infty} \int_{\mathbb{R}} \frac{(\log x)^n}{x^2} I_{[1,\infty]}(x)dx.$$

b.

$$\lim_{n\to\infty} \int_{\mathbb{R}} \frac{n^{3/2} x \sin x}{(n^2 + x^2)^2} dx.$$

Hint: First prove that corresponding functions are measurable for each $n \in \mathbb{N}$.

Solution: a. Answer: $+\infty$. For each $n \in \mathbb{N}$ the function $f_n : [1, \infty] \to \mathbb{R}$ given by

$$f_n(x) = \frac{(\log x)^n}{x^2}$$

is continuous on $[1, +\infty]$, so primage $f_n^{-1}(U)$ is a relatively open set in $[1, \infty]$ for each open set U in $\mathbb{R}$. This implies that all the functions are measurable and so the integrals are well-defined.

It is also clear that all the functions

$$\frac{(\log x)^n}{x^2} I_{[1,\infty]}(x)$$

are unsigned. Also by monotonicity of the integral we obtain that

$$\int_{\mathbb{R}} \frac{(\log x)^n}{x^2} I_{[1,\infty]}(x)dx \geq \int_{\mathbb{R}} \frac{(\log x)^n}{x^2} I_{[3,4]}(x)dx \geq \frac{(\log 3)^n}{16} \to \infty, \quad n \to \infty.$$

□

b. All the functions under the integral are measurable since they are continuous. Clearly, for each $x \in \mathbb{R}$

$$\lim_{n\to\infty} \frac{n^{3/2} x \sin x}{(n^2 + x^2)^2} = 0.$$

Now we will find absolutely integrable function that dominates all f_n, then we will get by dominated convergence theorem that

$$\lim_{n\to\infty} \int_{\mathbb{R}} \frac{n^{3/2} x \sin x}{(n^2 + x^2)^2} dx = 0.$$

For each $x \in \mathbb{R}$ we have for all $n \in \mathbb{N}$ that

$$\left| \frac{n^{3/2} x \sin x}{(n^2 + x^2)^2} \right| \leq \frac{n^{3/2}|x|}{(n^2 + x^2)^2} < |x|.$$

Also, for each $x \in \mathbb{R}$ and $n \in \mathbb{N}$

$$\left| \frac{n^{3/2} x \sin x}{(n^2 + x^2)^2} \right| \leq \frac{n^{3/2}|x|}{(n^2 + x^2)^2} \leq |x|^{-3/2}.$$

Indeed, this is equivalent to the inequality $n^{3/2}|x|^{5/2} \leq (n^2 + x^2)^2$. The last inequality is evident, since $n^{3/2}|x|^{5/2} \leq n^4$ for $|x| \leq n$, and $n^{3/2}|x|^{5/2} \leq x^4$ for $|x| \geq n$. So we can choose $g(x) = |x|$ for $|x| \leq 1$ and $g(x) = |x|^{-3/2}$ for $|x| > 1$. Clearly, g is continuous and therefore measurable and also $g \in L_1(\mathbb{R})$ because

$$\int_1^\infty x^{-3/2} dx < +\infty.$$

□

Problem 5. Let

$$f_n(x) = \frac{n^{3/2} x \sin x}{(n^2 + x^2)^2}.$$

a. Find the function $g(x)$ such that $\lim_{n\to\infty} f_n(x) = g(x)$ for all $x \in \mathbb{R}$.

b. Determine if $f_n \to g$ uniformly, in L_∞, almost uniformly, in L_1, or in measure.

c. Determine if there is $g \in L_1(\mathbb{R})$ such that $|f_n(x)| < g(x)$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$.

Solution: a. Answer: g(x)=0, see 4.b.
c. Answer: There is. For example, $g(x) = |x|$ for $|x| \le 1$ and $g(x) = |x|^{-3/2}$ for $|x| > 1$, see 4.b.
b. The inequality $n^2 + x^2 \ge 2|x|n$ implies that

$$\left|\frac{n^{3/2} x \sin x}{(n^2+x^2)^2}\right| \le \frac{n^{3/2}|x|}{(n^2+x^2)^2} \le \frac{n^{1/2}}{(n^2+x^2)} \le n^{-3/2}.$$

Therefore $f_n(x) \to 0$ uniformly and therefore in L_∞ and almost uniformly. Now it is easy to see that in 4.b. we actually proved that

$$\lim_{n\to\infty} \int_{\mathbb{R}} \left|\frac{n^{3/2} x \sin x}{(n^2+x^2)^2}\right| dx = 0,$$

hence $f_n(x) \to 0$ in L_1 and hence in measure. □