NTNU
Norwegian University of Science and Technology

Department of Mathematical Sciences

Examination paper for **TMA4225 Foundations of Analysis**

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Examination date: 12 December

Examination time (from–to): 09:00–13:00

Permitted examination support material: You need nothing but a pen/pencil, your head and a good mood!

Other information:
The exam contains 12 questions. Each solution will be graded as *rudimentary* (F), *acceptable* (D), *good* (C) or *excellent* (A). Five acceptable solutions guarantee an E; seven acceptable with at least one good a D; seven acceptable with at least five good a C; nine good with at least two excellent a B; nine good with at least seven excellent an A. These are guaranteed limits. Beyond that, the grade is based on the total achievement. Good luck!

Language: English

Number of pages: 2

Number of pages enclosed: 0

Checked by:

Informasjon om trykking av eksamensoppgave	
Originalen er:	
1-sidig ☐	**2-sidig** ☒
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Date Signature

Problem 1.
a. Prove that there is an open set E in $\mathbb{R}$ such that the following conditions hold simultaneously:
(i) $E \supset Q$, where Q is a set of all rationals;
(ii) $E \supset [0, 1]$;
(iii) $m(E) < 2$, where m is a Lebesgue measure.

b. Prove that for each measurable set A in $\mathbb{R}$ the function
$f(x) := m(A \cap [0, x])$ is well-defined and continuous on $[0, \infty)$.

c*. Prove that there is an open set E in $\mathbb{R}$ such that the following conditions hold simultaneously:
(i) $E \supset Q$;
(ii) $m(E) = 1$.

Problem 2.
a. Prove that the function

$$f(x) = \sum_{n=1}^{\infty} e^{-2^n x} I_{[0,\infty]}(x)$$

is unsigned measurable. Here $I_B(x)$ stands for the indicator function of a set $B \subset \mathbb{R}$.

b. Find the set of all $x \in \mathbb{R}$ for which $f(x) < +\infty$.

c. Find the Lebesgue integral

$$\int_{\mathbb{R}} f(x)dx.$$

Problem 3. Let $\mathcal{B}$ be a null σ-algebra on $\mathbb{R}$, that is the σ-algebra that consists of the sets of Lebesgue measure 0 and their complements. Prove that $\mathcal{B}$ is not atomic.

Problem 4. Find the following limits

a.

$$\lim_{n\to\infty}\int_{\mathbb{R}}\frac{(\log x)^n}{x^2}I_{[1,\infty]}(x)dx.$$

b.

$$\lim_{n\to\infty}\int_{\mathbb{R}}\frac{n^{3/2}x\sin x}{(n^2+x^2)^2}dx.$$

Hint: First prove that corresponding functions are measurable for each $n \in \mathbb{N}$.

Problem 5. Let

$$f_n(x) = \frac{n^{3/2}x\sin x}{(n^2+x^2)^2}.$$

a. Find the function $g(x)$ such that $\lim_{n\to\infty} f_n(x) = g(x)$ for all $x \in \mathbb{R}$.

b. Determine if $f_n \to g$ uniformly, in L_∞, almost uniformly, in L_1, or in measure.

c. Determine if there is $g \in L_1(\mathbb{R})$ such that $|f_n(x)| < g(x)$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$.