Solutions

Problem 1.

a. Prove that any one point subset $\{x\}$ of $\mathbb{R}$ is a countable intersection of open sets (G_δ set).

b. Let

$$A = \bigcup_{n=1}^{\infty} \left[n, n + \frac{1}{n^2 + n}\right].$$

Prove that A is a G_δ set. Find $m(A)$ the Lebesgue measure of A.

Solution: a. For any $x \in \mathbb{R}$ we have $\{x\} = \cap_{n=1}^{\infty}(x - 1/n, x + 1/n)$. Therefore $\{x\}$ is a G_δ set.

b. Similarly

$$A = \bigcap_{m=1}^{\infty} A_m,$$

where

$$A_m = \bigcup_{n=1}^{\infty} \left(n - \frac{1}{m}, n + \frac{1}{n^2 + n} + \frac{1}{m}\right).$$

Since the segments

$$\left[n, n + \frac{1}{n^2 + n}\right], \quad n \in \mathbb{N},$$

are pairwise disjoint we then have that

$$m(A) = \sum_{n=1}^{\infty} \frac{1}{n^2 + n} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1.$$

Problem 2.

a. Give definition of an unsigned measurable function $f : \mathbb{R} \to \mathbb{R}_+$.

b. Prove that the function

$$f(x) = \sum_{n=1}^{\infty} \frac{x}{n^2 + 1} I_{[n,n+1]}(x)$$

is measurable. Here $I_B(x)$ stands for the indicator function of a set $B \subset \mathbb{R}$.

c. Find the Lebesgue integral

$$\int_{\mathbb{R}} f(x)dx.$$

d. Prove that an unsigned function $f : \mathbb{R} \to \mathbb{R}_+$ that takes only *rational* values is measurable if and only if for each $q \in \mathbb{Q}$, the preimage $f^{-1}(\{q\})$ is a measurable set in $\mathbb{R}$.

Solution: a. An unsigned function $f : \mathbb{R} \to \mathbb{R}_+$ is measurable if and only if there exist a sequence of unsigned simple functions $f_n(x)$ such that

$$f(x) = \lim_{n\to\infty} f_n(x)$$

for all $x \in \mathbb{R}$.
b. Since f has countable number of discontinuity points ($\mathbb{N}$) then f is measurable.
c. Consider the following sequence of simple functions

$$f_m(x) = \sum_{n=1}^{m} \frac{n}{n^2+1} I_{[n,n+1]}(x).$$

It is clear that $f_m(x) \leq f(x)$ for all $x \in \mathbb{R}$ and $m \in \mathbb{N}$. Therefore by the definition of Lebesgue integral

$$\int_{\mathbb{R}} f(x)dx \geq Simp \int_{\mathbb{R}} f_m(x)dx = \sum_{n=1}^{m} \frac{n}{n^2+1}$$

for all $m \in \mathbb{N}$. Finally, since the series

$$\sum_{n=1}^{\infty} \frac{n}{n^2+1}$$

is divergent we obtain that

$$\int_{\mathbb{R}} f(x)dx = +\infty.$$

d. Assume that f is measurable. Then for each $q \in \mathbb{Q}$, $f^{-1}(\{q\})$ is measurable say as a preimage of a closed set (see Lemma 1.3.9 from Tao's book). On the other hand, if $f^{-1}(\{q\})$ are measurable for all $q \in \mathbb{Q}$, then for each $\lambda \in \mathbb{R}$

$$f^{-1}((\lambda, +\infty)) = \cup_{q>\lambda} f^{-1}(\{q\})$$

is measurable as a countable union of measurable set. Then we get the statement again by Lemma 1.3.9.

Problem 3. Let $(X, \mathcal{B}, \mu)$ be a measure space and let $f : X \to \mathbb{R}$ be an absolutely integrable function.

a. For every $n \geq 1$ define a set

$$E_n := \{x \in X : |f(x)| \geq n^{3/2}\}.$$

Prove that

$$\sum_{n=1}^{\infty} \mu(E_n) < \infty.$$

b. Denote similarly

$$B_n := \{x \in X : |f(x)| \geq n\}.$$

Do we always have that

$$\sum_{n=1}^{\infty} \mu(B_n) < \infty?$$

Solution: a. Since f is absolutely integrable then there exist $M > 0$ such that

$$\int_X |f| d\mu < M.$$

Now by Markov's inequality

$$\mu(E_n) < n^{-3/2} \int_X |f| d\mu < M n^{-3/2}.$$

Therefore

$$\sum_{n=1}^{\infty} \mu(E_n) < M \sum_{n=1}^{\infty} n^{-3/2} < +\infty.$$

b. Yes we always do! But the proof is slightly more tricky.
Let $C_n = \{x : |f(x)| \in [n, n+1)\}$ and $a_n = \mu(C_n)$. Clearly all C_n are disjoint. Now by approximation by simple functions we have that

$$\sum_{n=1}^{\infty} n a_n < M.$$

On the other hand, $B_n = \cup_{k \geq n} C_k$, so

$$\sum_{n=1}^{\infty} \mu(B_n) = \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} a_k = \sum_{n=1}^{\infty} n a_n < M,$$

where in the last identity we simply used double summation.

Problem 4. Find the following limits

a.
$$\lim_{n\to\infty}\int_{\mathbb{R}} x^{2n+1}e^{-n^2x^2}dx.$$

b.
$$\lim_{n\to\infty}\int_0^1 e^{-n\sin x}dx.$$

Hint: First prove that corresponding functions are measurable.

Solution: a. All functions $f_n(x) = x^{2n+1}e^{-n^2x^2}$ are measurable since they are continuous. Also, all f_n are absolutely integrable because

$$|x|^{2n+1}e^{-x^2} < \frac{1}{x^2}$$

for all large enough x depending on n and

$$\int_1^{\infty}\frac{1}{x^2}dx < +\infty.$$

Therefore by vertical truncation property (Exercise 1.3.10) for each fixed $n \in \mathbb{N}$

$$\int_{\mathbb{R}} x^{2n+1}e^{-n^2x^2}dx = \lim_{M\to\infty}\int_{-M}^{M} x^{2n+1}e^{-n^2x^2}dx = 0.$$

In the last identity we used that the Riemann integral of odd function over symmetric interval is 0. Hence

$$\lim_{n\to\infty}\int_{\mathbb{R}} x^{2n+1}e^{-n^2x^2}dx = 0.$$

b. All functions

$$f_n(x) = e^{-n\sin x}I_{[0,1]}(x).$$

are measurable since they are continuous in every point except 0 and 1. Now

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 1, x = 0, \\ 0,\ x \in \mathbb{R}\setminus\{0\}. \end{cases}$$

Also,

$$|f_n(x)| \le I_{[0,1]}(x)$$

for all $x \in \mathbb{R}$. Hence by dominated convergence theorem

$$\lim_{n\to\infty} \int_0^1 e^{-n\sin x} dx = 0.$$

Problem 5. Let

$$f_n(x) = e^{-n\sin x} I_{[0,1]}(x).$$

a. Find the function $g(x)$ such that $\lim_{n\to\infty} f_n(x) = g(x)$ for all $x \in \mathbb{R}$.

b. Determine if $f_n \to g$ uniformly, in L_∞, almost uniformly, in L_1, or in measure.

Solution: a. As we see from before

$$g(x) = \begin{cases} 1, x = 0, \\ 0, \ x \in \mathbb{R} \setminus \{0\}. \end{cases}$$

b. Since each f_n is continuous on $(0,1)$ then for each set B of measure 0 we have

$$b_n := \sup_{\mathbb{R}\setminus B} |f_n(x) - g(x)| = f_n(0) = 1.$$

Therefore the sequence f_n doesn't converge to g in L_∞ or uniformly. On the other hand, for each $\epsilon \in (0,1)$

$$c_{n,\epsilon} := \sup_{\mathbb{R}\setminus[0,\epsilon)} |f_n(x) - g(x)| = f_n(\epsilon) = e^{-n\sin\epsilon} \to 0, \quad n \to \infty.$$

Therefore the sequence f_n converges to g almost uniformly and hence in measure. Finally, it follows from Problem 4, part b, that f_n converges to g in L_1.