NTNU
Norwegian University of Science and Technology

Department of Mathematical Sciences

Examination paper for **TMA4225 Foundations of Analysis**

Academic contact during examination: Andrii Bondarenko

Phone: 46341811

Examination date: 2 December

Examination time (from–to): 09:00–13:00

Permitted examination support material: You need nothing but a pen/pencil, your head and a good mood!

Other information:
The exam contains 12 questions. Each solution will be graded as *rudimentary* (F), *acceptable* (D), *good* (C) or *excellent* (A). Five acceptable solutions guarantee an E; seven acceptable with at least one good a D; seven acceptable with at least five good a C; nine good with at least two excellent a B; nine good with at least seven excellent an A. These are guaranteed limits. Beyond that, the grade is based on the total achievement. Good luck!

Language: English

Number of pages: 2

Number of pages enclosed: 0

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Informasjon om trykking av eksamensoppgave	
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Date Signature

Problem 1.
a. Prove that any one point subset $\{x\}$ of $\mathbb{R}$ is a countable intersection of open sets (G_δ set).

b. Let

$$A = \bigcup_{n=1}^{\infty} \left[n, n + \frac{1}{n^2+n}\right].$$

Prove that A is a G_δ set. Find $m(A)$ the Lebesgue measure of A.

Problem 2.
a. Give definition of an unsigned measurable function $f : \mathbb{R} \to \mathbb{R}_+$.

b. Prove that the function

$$f(x) = \sum_{n=1}^{\infty} \frac{x}{n^2+1} I_{[n,n+1]}(x)$$

is measurable. Here $I_B(x)$ stands for the indicator function of a set $B \subset \mathbb{R}$.

c. Find the Lebesgue integral

$$\int_{\mathbb{R}} f(x)dx.$$

d. Prove that an unsigned function $f : \mathbb{R} \to \mathbb{R}_+$ that takes only *rational* values is measurable if and only if for each $q \in \mathbb{Q}$, the preimage $f^{-1}(\{q\})$ is a measurable set in $\mathbb{R}$.

Problem 3. Let $(X, \mathcal{B}, \mu)$ be a measure space and let $f : X \to \mathbb{R}$ be an absolutely integrable function.
a. For every $n \geq 1$ define a set

$$E_n := \{x \in X : |f(x)| \geq n^{3/2}\}.$$

Prove that

$$\sum_{n=1}^{\infty} \mu(E_n) < \infty.$$

b. Denote similarly

$$B_n := \{x \in X : |f(x)| \geq n\}.$$

Do we always have that

$$\sum_{n=1}^{\infty} \mu(B_n) < \infty?$$

Problem 4. Find the following limits

a.

$$\lim_{n\to\infty} \int_{\mathbb{R}} x^{2n+1} e^{-n^2x^2} dx.$$

b.

$$\lim_{n\to\infty} \int_0^1 e^{-n\sin x} dx.$$

Hint: First prove that corresponding functions are measurable.

Problem 5. Let

$$f_n(x) = e^{-n\sin x} I_{[0,1]}(x).$$

a. Find the function $g(x)$ such that $\lim_{n\to\infty} f_n(x) = g(x)$ for all $x \in \mathbb{R}$.

b. Determine if $f_n \to g$ uniformly, in L_∞, almost uniformly, in L_1, or in measure.