ALL HOMEWORK ASSIGNMENTS FOR TMA4225-FOUNDATIONS OF ANALYSIS-FALL 2015

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1. Introduction

This document collects in one place all the homework sets assigned throughout the semester.

To prepare for the final exam, review your notes and homework problems carefully.

We have used Terence Tao's book "An Introduction to Measure Theory". A pdf file of the book is freely available at `https://terrytao.files.wordpress.com/2011/01/measure-book1.pdf`.

We have covered most but not all of the topics in the main chapter of the book: Chapter 1. Measure Theory. Many important statements (theorems, lemmas, propositions) in Tao's book are listed as exercises. We proved them in class or, in some cases, they were assigned as homework. Therefore, while the book is a good reference for your study, the best reference should be your own handwritten notes from lectures and exercise sessions.

2. Second week homework assignment

Problem 2.1. Prove that the following sets are countable:

(a) The set of strictly positive integers $\mathbb{N}^\star = \{1, 2, \dots n, \dots\}$.
(b) The set of all even natural numbers.
(c) The set of all odd natural numbers.
(d) The set $\mathbb{Z}$ of all integers.

Problem 2.2. Prove that all (proper) intervals have the same cardinality, which we call the *continuum* cardinality.

Problem 2.3. [*] Prove Cantor-Schröder-Bernstein's theorem:

If $\operatorname{card} A \le \operatorname{card} B$ and $\operatorname{card} B \le \operatorname{card} A$ then $\operatorname{card} A = \operatorname{card} B$.

In other words, if there are a one-to-one function $f\colon A \to B$ and a one-to-one function $g\colon B \to A$, then there is a *bijection* $h\colon A \to B$.

Follow the steps in exercise 1.36 in the textbook.

Problem 2.4. Prove the following:

(a) Any *finite* subset of $\mathbb{R}$ is a closed set.
(b) $\mathbb{N}$ and $\mathbb{Z}$ are closed sets.
(c) $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are not closed sets.
(d) None of the sets above is open (except for the empty set).
(e) If $P(x)$ is a polynomial, then $\{x \in \mathbb{R}\colon P(x) > 0\}$ is open.

Problem 2.5. [*] Prove that any open set can be written as a disjoint union of at most countably many open intervals.

Problem 2.6. Prove that if F is a closed set, then its complement $F^{\complement} = \mathbb{R} \setminus F$ is an open set.

Problem 2.7. Prove (*without* using the Heine-Borel theorem) that if K is a compact set and if F is a closed set such that $F \subset K$, then F is a compact set as well.

Problem 2.8. Prove that any decreasing sequence of *closed* intervals whose lengths converge to 0 has a nonempty intersection.

In other words: let $I_1,\ I_2, \dots, I_n,\ I_{n+1}, \dots$ be closed intervals so that

$$I_1 \supset I_2 \supset \dots \supset I_n \supset I_{n+1} \supset \dots$$

and $\left|I_n\right| \to 0$ as $n \to \infty$.

Prove that

$$\bigcap_{n \ge 1} I_n \neq \emptyset\,.$$

This problem may be divided into the following easier steps.

(a) Pick in each interval I_n a point x_n and show that the sequence $\{x_n\}$ thus obtained is Cauchy, hence convergent.

(b) Let x be the limit of this sequence. Prove that x belongs to all intervals I_n (use the fact that closed intervals are closed sets).
(c) Prove that in fact $\bigcap_{n \geq 1} I_n$ only contains the point x.

Problem 2.9. Prove that if $f \colon \mathbb{R} \to \mathbb{R}$ is a continuous function and $c \in \mathbb{R}$, then

(a) The set $\{x \in \mathbb{R} \colon f(x) > c\}$ is open.
(b) The set $\{x \in \mathbb{R} \colon f(x) < c\}$ is open.
(c) The set $\{x \in \mathbb{R} \colon f(x) \in (c, d)\}$ is open.
(d) The set $\{x \in \mathbb{R} \colon f(x) = c\}$ is closed.
(e) The set $\{x \in \mathbb{R} \colon f(x) \leq c\}$ is closed.

3. Third week homework assignment

Problem 3.1. Prove that the elementary measure is monotone and sub-additive. In other words, show that the map $m \colon \mathcal{E}(\mathbb{R}^d) \to \mathbb{R}$ satisfies the following:

(i) For all elementary sets E and F, if $E \subset F$ then $m(E) \leq m(F)$.
(ii) For all elementary sets E and F, $m(E \cup F) \leq m(E) + m(F)$.

Explain why the same properties hold for the Jordan measure.

Problem 3.2. Prove that the Cantor set is Jordan measurable and that its Jordan measure is 0.

Problem 3.3. Prove that if E and F are Jordan measurable sets, then $E \cap F$, $E \setminus F$ and $E \triangle F$ are Jordan measurable as well.

Hint: Use the characterization of Jordan measurability in terms of elementary sets that approximate from within and from without. Then everything reduces to some (Boolean) operations with sets.

Perhaps the trickiest part is showing that $E \setminus F$ is Jordan measurable. You may use (after verifying that they are indeed true) the following relations between sets.

If $A \subset E \subset B$ and $C \subset F \subset D$, then

$$A \setminus D \subset E \setminus F \subset B \setminus C.$$

Moreover,

$$(B \setminus C) \setminus (A \setminus D) \subset (B \setminus A) \cup (D \setminus C).$$

Problem 3.4. Prove that a bounded set $E \subset \mathbb{R}^d$ is Jordan measurable if and only if for all $\epsilon > 0$ there is an elementary set A such that $m^{*,J}(E \triangle A) < \epsilon$.

Problem 3.5. Let $E \subset \mathbb{R}^d$ be any bounded set. Prove the following:

(a) $m^{*,J}(\overline{E}) = m^{*,J}(E)$, where $\overline{E}$ denotes the closure of E.
(b) $m_{*,J}(\mathring{E}) = m_{*,J}(E)$, where $\mathring{E}$ denotes the interior of E.
(c) E is Jordan measurable if and only if $m^{*,J}(\partial E) = 0$, where ∂E denotes the boundary of E.

Hint: During the exercise section, someone made me aware of the fact that part (c) is quite tricky. The difficult part is to show that if $m^{*,J}(\partial E) = 0$ then E is Jordan measurable. Here is a hint.

Since $m^{*,J}(\partial E) = 0$, for every $\epsilon > 0$ there is an elementary set D with $\partial E \subset D$ and $m(D) < \epsilon$. We may assume that D is an open set (why?). Then $\overline{E} \setminus D$ is *compact* (why?).

Note that $\overline{E} \setminus D \subset \mathring{E}$, and by compactness we can find an elementary set B such that

$$\overline{E} \setminus D \subset B \subset \mathring{E}.$$

This implies $\overline{E} \subset B \cup D$. Since $B \cup D$ is an elementary set, we may then derive that $m^{*,J}(\overline{E}) \le m_{*,J}(\mathring{E}) + \epsilon$. Let $\epsilon \to 0$, then use parts (a) and (b) to conclude that E is Jordan measurable.

This is of course a rough sketch of the proof, you have many details to fill in. Try it till next week on your own, and then I will post the detailed solution.

Problem 3.6. Let $E := \mathbb{Q} \cap [0, 1]$. Prove the following

(a) $m_{*,J}(E) = 0$.
(b) $m^{*,J}(E) = 1$.

Conclude that E is not Jordan measurable.

Problem 3.7. Prove that if f and g are Darboux integrable functions on $[a, b]$, then $f + g$ is Darboux integrable as well and

$$\int_a^b (f + g) = \int_a^b f + \int_a^b g.$$

Hint: Use the characterization of the Darboux integrability in terms of "good" step functions (that approximate from below and above).

Problem 3.8. We have shown in class that if E is a Jordan measurable subset of $\mathbb{R}$, then its indicator function $\mathbf{1}_E(x)$ is Darboux integrable.

Prove that the reverse is also true. That is, prove that if the indicator function $\mathbf{1}_E(x)$ of some set E is Darboux integrable, then the set E must be Jordan measurable.

4. Fourth week homework assignment

Problem 4.1. Show that a countable union of Jordan measurable sets may not be Jordan measurable.

Then show that a countable intersection of Jordan measurable sets may not be Jordan measurable.

Hint: Look for an example of a set that is not Jordan measurable, but which you can break down into countably many Jordan measurable subsets. For the second problem, on intersection, consider the complements of these subsets relative to some large enough box.

Problem 4.2. Finish up the proof of the theorem discussed in class saying that if a sequence $\{f_n\}_{n\geq 1}$ of Riemann integrable functions on $[a, b]$ converges *uniformly* to f, then f is Riemann integrable as well and

$$\int_a^b f(x)dx = \lim_{n\to\infty} \int_a^b f_n(x)dx.$$

Problem 4.3. Let $\mathcal{X}$ and $\mathcal{Y}$ be subsets of $\mathbb{R}$ such that $\mathcal{X} \subset \mathcal{Y}$. Prove that

$$\inf \mathcal{X} \geq \inf \mathcal{Y} \quad \text{and} \ \sup \mathcal{X} \leq \sup \mathcal{Y}.$$

Problem 4.4. Prove that the Jordan outer measure of a set $E \subset \mathbb{R}^d$ is the infimal cost required to cover E by a finite union of boxes.

More precisely, prove that

$$m^{*,J}(E) = \inf \{ \sum_{i=1}^{k} \big|B_i\big| \ : \ E \subset B_1 \cup B_2 \ldots \cup B_k, \\ \text{where } B_1, B_2, \ldots, B_k \text{ are boxes and } k \geq 1\}.$$

Note: Finite unions of boxes are elementary sets. So the only difference between what is written above and the definition of the Jordan outer measure is that in the formula above we do *not* require the boxes that cover E to be *disjoint*. In other words, what the problem is saying is that it does not matter if we throw in all boxes that cover E; whether they are disjoint or not, when we take the infimum we get the same quantity.

Problem 4.5. Prove that a bounded set E is Jordan measurable if and only if it is "almost" elementary, in the sense that: for any $\epsilon > 0$ there is an elementary set B such that $E \subset B$ and $m^{*,J}(B \setminus E) < \epsilon$.

Problem 4.6. Find a function $f\colon [0, 1] \to \mathbb{R}$ that has an *uncountable* set of discontinuities, but which is still Riemann integrable.

Hint: Use Cantor's set.

Problem 4.7. Prove that if a set $E \subset \mathbb{R}^d$ has Lebesgue *outer* measure zero (we may call such sets *negligible*), then E is Lebesgue measurable.

Problem 4.8. Prove that the Lebesgue outer measure is monotone: if $E \subset F \subset \mathbb{R}^d$, then $m^*(E) \le m^*(F)$.

Conclude, using the previous problem, that any subset of a negligible set is Lebesgue measurable and it has Lebesgue measure zero.

Problem 4.9. Let E be a bounded set in $\mathbb{R}^d$.

(a) Prove that if E is Jordan measurable, then it is Lebesgue measurable as well.
(b) [*] You may try to prove that in this case, the Jordan measure and the Lebesgue measure of E are the same (this is not easy, and we will do it in class anyway).

Hint (for part (a)): Use the fact discussed in class, saying that if we pay just ϵ more, we can always consider *open* boxes (rather than just any boxes) to cover our sets E.

5. Fifth week homework assignment

Problem 5.1. Prove the translation invariance of the Lebesgue outer measure. That is, show that if $E \subset \mathbb{R}^d$ and $x \in \mathbb{R}^d$ then

$$m^*(x + E) = m^*(E).$$

Problem 5.2. Prove a similar property for multiplication on $\mathbb{R}$: if $E \subset \mathbb{R}$ and $x \in \mathbb{R}$, if we denote $x\,E := \{x \cdot a \colon a \in E\}$, then

$$m^*(x\,E) = \left|x\right| m^*(E).$$

Problem 5.3. Let $E \subset \mathbb{R}$ with $m^*(E) < \infty$.
Define the function $f \colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) := m^*(E \cap (-\infty, x]).$$

Prove that f is uniformly continuous on $\mathbb{R}$.

Problem 5.4. Let $f \colon \mathbb{R} \to \mathbb{R}$ be differentiable at all points.

(a) Prove that if for all $x \in \mathbb{R}$ we have $\left|f'(x)\right| \le 1$, then for every subset $E \subset \mathbb{R}$ we have

$$m^*(f(E)) \le m^*(E).$$

We say that the function f *contracts* the measure.
Hint: Use the mean value theorem from calculus.

(b) Find an example of a differentiable function f such that for some x, $\left|f'(x)\right| > 1$, and which does not contract the measure.
Hint: Use Problem 2 above.

Problem 5.5. Find an example of a set $E \subset \mathbb{R}$ for which

$$m^*(E) > \sup \{m^*(U) \colon U \subset E, U \text{ is open}\}.$$

This will show that the *exact* inner regularity analogue of the outer regularity is false.

Problem 5.6. Prove that if $B_1, B_2, \ldots, B_N$ are *almost* disjoint boxes (meaning that their interiors are disjoint) then

$$m\,(\bigcup_{k=1}^{N} B_k) = \sum_{k=1}^{N} \left| B_k \right| .$$

The measure m above refers to the Jordan measure (since a union of boxes is elementary), which of course coincides with the Lebesgue outer measure.

It was mentioned in class that the *additivity* (finite of countable) of the Lebesgue outer measure does *not* always hold. However, we can show that it does hold in many interesting cases.

Let us first remember the following useful result.

Lemma. If $E = \bigcup_{n=1}^{\infty} B_n$, where B_n are almost disjoint boxes, then

$$m^*(E) = \sum_{n=1}^{\infty} \left| B_n \right| .$$

Moreover, any *open* set can be expressed in this way, as a countable union of almost disjoint boxes.

We are now ready for the first result on the additivity of the Lebesgue outer measure.

Problem 5.7. Prove that if U and V are *open* sets and if $U \cap V = \emptyset$, then

$$m^*(U \cup V) = m^*(U) + m^*(V).$$

Hint: Use the lemma above for both U and V.

The next result on the additivity of the Lebesgue outer measure is for compact sets. They require a bit more preparation. We begin with a definition.

Definition. If A and B are two subsets of $\mathbb{R}$ (or of $\mathbb{R}^d$), we define the *distance* between them as the infimal distance between a point in A and a point in B. More precisely,

$$\operatorname{dist}(A, B) := \inf \{\left|x - y\right| \colon x \in A, y \in B\}.$$

Problem 5.8. Prove that if K and L are compact sets in $\mathbb{R}$ (or in $\mathbb{R}^d$), and if $K \cap L = \emptyset$, then

$$\operatorname{dist}(K, L) > 0\,.$$

Hint: Assume by contradiction that $\operatorname{dist}(K, L) = 0$.
At some point you may also need to use the fact that any sequence in a compact set has a convergent subsequence.

Problem 5.9. Prove that if A and B are two subsets of $\mathbb{R}$ (or of $\mathbb{R}^d$) such that $\operatorname{dist}(A, B) > 0$, then there are two *open* sets U and V such that $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$.

Hint: "Thicken" (like we did before in class) each of the sets A and B by a small enough amount - how small will depend on $\operatorname{dist}(A, B)$.

Problem 5.10. Prove that if K and L are compact sets in $\mathbb{R}$ (or in $\mathbb{R}^d$), and if $K \cap L = \emptyset$, then

$$m^*(K \cup L) = m^*(K) + m^*(L)\,.$$

Hint: Use problem 8, then 9 and then 7.
Also use the fact (proven in class) that for any set E,

$$m^*(E) = \inf\{m^*(U) \colon E \subset U \text{ and } U \text{ is open}\}.$$

6. Sixth week homework assignment

Problem 6.1. Prove that if $E_1, E_2, \dots$ is a countable sequence of Lebesgue measurable sets, then $\bigcap_{n=1}^{\infty} E_n$ is also Lebesgue measurable.

Hint: Use one of de Morgan's rules and the analogue result proven in class for countable unions of measurable sets.

Problem 6.2. Show that for any sets $F \subset E \subset \mathbb{R}^d$ we have

$$m^*(E) - m^*(F) \le m^*(E \setminus F).$$

Hint: Use the sub-additivity of the Lebesgue outer measure.

Problem 6.3. Let $E \subset \mathbb{R}^d$ be a any set. Show that there is a Lebesgue *measurable* set E' such that $E \subset E'$ and $m^*(E) = m(E')$.

Hint: Use the outer regularity of the Lebesgue outer measure (approximation by open sets from without).

The goal of the next problem is to prove several *equivalent* ways of describing Lebesgue measurability.

Problem 6.4. Let $E \subset \mathbb{R}^d$. Prove that the following are equivalent:

(1) E is Lebesgue measurable, meaning: for all $\epsilon > 0$ there is U open such that $E \subset U$ and $m^*(U \setminus E) < \epsilon$.
(2) E is "almost open" in the following sense: for all $\epsilon > 0$ there is U open such that $m^*(E \triangle U) < \epsilon$.
(3) E has a good "inner approximation by closed sets": for all $\epsilon > 0$ there is F closed such that $F \subset E$ and $m^*(E \setminus F) < \epsilon$.
(4) E is "almost closed" in the following sense: for all $\epsilon > 0$ there is F closed such that $m^*(E \triangle F) < \epsilon$.

Hint: Similar to the analogue statements for Jordan measurable sets.

Problem 6.5. Prove the following inner regularity property of the Lebesgue measure. If $E \subset \mathbb{R}^d$ is Lebesgue measurable, then

$$m(E) = \sup \{m(K) \colon K \text{ is compact, } K \subset E\}.$$

Hint: First assume that E is bounded and use the "inner approximation by closed sets" characterization of measurable sets.

If E is unbounded, consider its intersection with larger and larger boxes and use the monotone convergence theorem for sets.

The goal of the next problem is to characterize in different (equivalent) ways Lebesgue measurable sets of *finite* measure.

Problem 6.6. Let $E \subset \mathbb{R}^d$. Prove that the following are equivalent:

(1) E is Lebesgue measurable and $m(E) < \infty$.
(2) "approximation by open with finite measure": for all $\epsilon > 0$ there is U open with $m(U) < \infty$ such that $E \subset U$ and $m^*(U \setminus E) < \epsilon$.
(3) "almost open and bounded": for all $\epsilon > 0$ there is U open and bounded such that $m^*(E \triangle U) < \epsilon$.
(4) "inner approximation by compact": for all $\epsilon > 0$ there is K compact such that $K \subset E$ and $m^*(E \setminus K) < \epsilon$.
(5) "almost compact": for all $\epsilon > 0$ there is K compact such that $m^*(E \triangle K) < \epsilon$.
(6) "almost measurable and bounded": for all $\epsilon > 0$ there is S measurable and bounded such that $m^*(E \triangle S) < \epsilon$.
(7) "almost finite measure": for all $\epsilon > 0$ there is S measurable with $m(S) < \infty$ such that $m^*(E \triangle S) < \epsilon$.

7. Seventh week homework assignment

Problem 7.1. Prove that if $E \subset \mathbb{R}^d$ is Lebesgue measurable, then for any subset S of $\mathbb{R}^d$ we have

$$m^*(S) = m^*(S \cap E) + m^*(S \setminus E).$$

Hint: We have already established this property in class for *bounded* sets. You only need to use the "larger and larger" boxes trick to extend this property to unbounded sets.

Problem 7.2. Prove that if s and σ are simple functions on $\mathbb{R}^d$, and if $c \in \mathbb{R}$, then $s + \sigma$, $s - \sigma$, $s \cdot \sigma$ and $c\, s$ are also simple functions.

Hint: The only tricky part is $s \cdot \sigma$. Use (after convincing yourselves that it is true) the following simple property of indicator functions:

$$\mathbf{1}_E \cdot \mathbf{1}_F = \mathbf{1}_{E \cap F}.$$

Problem 7.3. Prove that if s and σ are nonnegative simple functions on $\mathbb{R}^d$ and if $c \in [0, \infty]$, then

$$\int_{R^d} (s + \sigma)(x)\, dx = \int_{\mathbb{R}^d} s(x)\, dx + \int_{\mathbb{R}^d} \sigma(x)\, dx.$$

$$\int_{\mathbb{R}^d} c\, s(x)\, dx = c \int_{\mathbb{R}^d} s(x)\, dx.$$

8. Eighth week homework assignment

$\diamond$ The next few problems are about some basic properties (left unproven in class) of the integral of *simple, non-negative* functions.

Problem 8.1. Prove that if $f, g \colon \mathbb{R}^d \to [0, +\infty]$ are simple functions, and if $c \in [0, +\infty]$, then

$$\int_{\mathbb{R}^d} (f + g)(x)\, dx = \int_{\mathbb{R}^d} f(x)\, dx + \int_{\mathbb{R}^d} g(x)\, dx$$

$$\int_{\mathbb{R}^d} c\, f(x)\, dx = c \int_{\mathbb{R}^d} f(x)\, dx.$$

Problem 8.2. Let $f \colon \mathbb{R}^d \to [0, +\infty]$ be a simple function. Prove that if $f(x) < \infty$ a.e. and if $\operatorname{supp}(f)$ has finite measure then

$$\int_{\mathbb{R}^d} f(x)\, dx < \infty.$$

Problem 8.3. If $f, g\colon \mathbb{R}^d \to [0, +\infty]$ are simple and if $f = g$ a.e. then $\int f = \int g$.

$\diamond$ The next few problems are about basic properties of the integral of *simple* functions that may take both positive and negative values.

Problem 8.4. Prove that if $f, g\colon \mathbb{R}^d \to \mathbb{R}$ are simple functions, and if $c \in \mathbb{R}$, then

$$\int_{\mathbb{R}^d} c\, f(x)\, dx = c \int_{\mathbb{R}^d} f(x)\, dx.$$

Problem 8.5. If $f, g\colon \mathbb{R}^d \to \mathbb{R}$ are simple functions and if $f \le g$ a.e. then $\int f \le \int g$.

$\diamond$ The next few problems are about basic properties of *measurable, non-negative* functions.

Problem 8.6. If $f\colon \mathbb{R}^d \to [0, +\infty]$ is a Lebesgue measurable function and if $\phi\colon [0, \infty] \to [0, \infty]$ is *continuous*, then $\phi \circ f$ is Lebesgue measurable.

Problem 8.7. If $f, g\colon \mathbb{R}^d \to [0, +\infty]$ are Lebesgue measurable and if $c \in [0, \infty)$, then $f + g$, $c\, f$ and $f \cdot g$ are Lebesgue measurable.

Problem 8.8. Prove that $f\colon \mathbb{R}^d \to [0, \infty]$ is a simple function if and only if it is measurable and it only takes finitely many values.

9. Ninth week homework assignment

These problems are about some basic properties (left unproven in class) of the integral of *measurable, non-negative* functions.

Problem 9.1. Prove that if $f, g\colon \mathbb{R}^d \to [0, \infty]$ are Lebesgue measurable and if $f \le g$ a.e. then $\int f \le \int g$.

Note: Keep in mind that at this point, linearity has not been yet established. So the only thing you could use is the *definition* of the Lebesgue integral of a measurable, non-negative function.

Problem 9.2. Prove the remaining case from class of the "horizontal truncation" property:

Let $f\colon \mathbb{R}^d \to [0, \infty]$ be a Lebesgue measurable function, and for each integer $n \ge 1$ define its horizontal truncation $f_n(x) := \min\{f(x), n\}$. Prove that if $\int f = +\infty$ then

$$\lim_{n\to\infty} \int f_n = +\infty.$$

Problem 9.3. Prove the remaining case from class of the "vertical truncation" property:

Let $f\colon \mathbb{R}^d \to [0,\infty]$ be a Lebesgue measurable function, and for each integer $n \geq 1$ define its vertical truncation $f_n := f \cdot \mathbf{1}_{\{x\colon |x| \leq n\}}$. Prove that if $\int f < \infty$ then

$$\int f \leq \lim_{n\to\infty} \int f_n \,.$$

Note: I should explain what I mean by $|x|$.

If $d = 1$, $|x|$ is just the absolute value of x, so $\{x\colon\ |x| \leq n\} = [-n, n]$.

If $d > 1$, then write $x \in \mathbb{R}^d$ as $x = (x_1, x_2, \ldots, x_d)$, and define $|x| := \max\{|x_1|, |x_2|, \ldots, |x_d|\}$. So the set $\{x\colon\ |x| \leq n\} = [-n, n]^d$, which is a box in $\mathbb{R}^d$.

Problem 9.4. Prove that if $f\colon \mathbb{R}^d \to [0,\infty]$ is Lebesgue measurable and if $c \in [0, \infty)$, then

$$\int c\, f = c \int f \,.$$

The goal of the next few problems is to establish the *additivity* of the Lebesgue integral of *non-negative measurable* functions. That is, we will show that if $f, g\colon \mathbb{R}^d \to [0,\infty]$ are Lebesgue measurable, then

$$\int (f+g) = \int f + \int g \,.$$

We will accomplish this goal in a few steps.

Problem 9.5. Prove first the inequality

$$\int f + \int g \leq \int (f+g) \,.$$

Hint: Begin with a simple function *below* f and one below g, add up, use the additivity of the integral for *simple* functions, take supremums.

Problem 9.6. Assume that f and g "live in a box" (the same box). Prove the reverse inequality under this assumption:

$$\int (f+g) \leq \int f + \int g \,.$$

Hint: Begin with a simple function *above* f and one above g, add up, use the additivity of the integral for *simple* functions, take infimums.

Problem 9.7. Remove the assumption from the previous problem that f and g "live in a box". In other words, prove that for any measurable, nonnegative functions f and g, we have

$$\int (f+g) \leq \int f + \int g\,.$$

Hint: Truncate the functions f and g horizontally and vertically; the truncations will live in boxes, so the previous problem will apply to them; to conclude, use a limiting argument (namely the horizontal and vertical truncation properties).

At some point in the argument, it may help to know (prove this separately that if f_n, g_n and $(f+g)_n$ represent horizontal truncations of f, g and $f+g$ respectively, then

$$(f+g)_n \leq f_n + g_n.$$

10. Tenth week homework assignment

Problem 10.1. Prove that if $\{y_n\}_{n\geq 1} \subset \mathbb{R}$ is a sequence of real numbers and if $y_n \to y$ as $n \to \infty$, then

$$y_n^+ \to y^+ \quad \text{and} \quad y_n^- \to y^- \quad \text{as } n \to \infty.$$

Hint: Consider three separate cases: $y > 0$, $y < 0$ and $y = 0$.

Problem 10.2. Prove that $L^1(\mathbb{R}^d)$ is a normed space with the norm

$$\|f\|_{L^1} := \int_{\mathbb{R}^d} \big|f(x)\big|\,dx.$$

In other words:

(a) Show that the set of absolutely integrable functions is a subspace of the vector space of all functions on $\mathbb{R}^d$.
(b) Let $L^1(\mathbb{R}^d)$ be the space of absolutely integrable functions where two functions that are equal a.e. are identified (i.e. they are regarded as representing the same vector). Prove that $\|\cdot\|_{L^1}$ defined above satisfies the three axioms of a norm.

Problem 10.3. Prove that the Lebesgue integral is a linear map from $L^1(\mathbb{R}^d)$ to $\mathbb{R}$. In other words, show that

$$\int_{\mathbb{R}^d} (f+g) = \int_{\mathbb{R}^d} f + \int_{\mathbb{R}^d} g \quad \text{and} \int_{\mathbb{R}^d} cf = c\int_{\mathbb{R}^d} f,$$

for any absolutely integrable functions f and g and any constant $c \in \mathbb{R}$.

Hint: Use the same strategy employed to establish the linearity of the integral for simple *signed* functions, once the linearity was known for *unsigned* functions.

Problem 10.4. Prove the monotonicity of the integral. If $f, g \colon \mathbb{R}^d \to \mathbb{R}$ are absolutely integrable and if $f \le g$ a.e. then

$$\int_{\mathbb{R}^d} f \le \int_{\mathbb{R}^d} g.$$

Problem 10.5. Prove the following triangle inequality: if $f \colon \mathbb{R}^d \to \mathbb{R}$ is absolutely integrable then

$$\Big| \int_{\mathbb{R}^d} f \Big| \le \int_{\mathbb{R}^d} |f|.$$

Problem 10.6. Prove the divisibility property: if $f \colon \mathbb{R}^d \to \mathbb{R}$ is absolutely integrable and if $E \subset \mathbb{R}^d$ is measurable, then

$$\int_{\mathbb{R}^d} f = \int_E f + \int_{E^{\complement}} f\,,$$

where $E^{\complement} = \mathbb{R}^d \setminus E$ is the complement of E and if S is any measurable set, $\int_S f := \int_{\mathbb{R}^d} f \cdot \mathbf{1}_S$.

Problem 10.7. Prove that if $\{f_n\}_{n \ge 1}$ is a sequence of measurable functions on $\mathbb{R}^d$ and if for a.e. $x \in \mathbb{R}^d$ we have

$$\lim_{n \to \infty} f_n(x) = f(x),$$

then the limit function $f(x)$ is measurable as well.

Hint: Use the same approach we described in class when proving the equivalent notions of measurability (namely the idea of the proof of Lemma 1.3.9 (iii) implies (v) in Tao's book).

11. Eleventh week homework assignment

Problem 11.1. Prove that the sequence of functions $f_n \colon \mathbb{R} \to \mathbb{R}$,

$$f_n(x) = \frac{x}{n}$$

converges to the function $f(x) = 0$ locally uniformly but not uniformly.

Problem 11.2. Prove that the sequence of functions $f_n \colon \mathbb{R} \to \mathbb{R}$,

$$f_n(x) = \begin{cases} \frac{1}{n\,x} & \text{if } x > 0 \\ 0 & \text{if } x \le 0 \end{cases}$$

converges to the function $f(x) = 0$ pointwise but not locally uniformly.

Problem 11.3. Verify that an atomic algebra is a σ-algebra.

Here is a reminder of how an atomic algebra is defined. Let X be a set and consider a partition

$$X = \bigcup_{\alpha \in I} A_\alpha$$

into disjoint subsets (we refer to these subsets as "atoms"). Let

$$\mathcal{A} := \Big\{ \bigcup_{\alpha \in J} A_\alpha \colon J \subset I \Big\}.$$

In other words, $\mathcal{A}$ consists of all possible unions of atoms.

Then $\mathcal{A}$ is a σ-algebra.

Problem 11.4. Consider an atomic algebra with a *countable* set of atoms. Explain how one can define a measure on this measurable space and verify that what you have defined is indeed a measure.

Hint: Begin by assigning to each atom a certain "mass".

Problem 11.5. Prove that the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$ may be generated by either of the following collections of subsets of $\mathbb{R}^d$:

(a) All closed sets;
(b) All compact sets;
(c) All open balls;
(d) All boxes.

Problem 11.6. Let $(X, \mathcal{B}, \mu)$ be a measure space. Prove the monotone convergence theorem for sets:

(a) Upwards: if $E_n \nearrow E$ as $n \to \infty$, then $\mu(E_n) \to \mu(E)$ as $n \to \infty$.
(b) Downwards: if $E_n \searrow E$ as $n \to \infty$ and if $\mu(E_N) < \infty$ for some N then $\mu(E_n) \to \mu(E)$ as $n \to \infty$.

Problem 11.7. Let $(X, \mathcal{B})$ be a measurable space and let $f \colon X \to [0, \infty]$ be a function. Show that f is measurable if and only if all the level sets

$$\{x \in X \colon f(x) > \lambda\}$$

are $\mathcal{B}$-measurable.

Recall that we have defined f to be *measurable* if for every open set $U \subset [0, \infty]$, its pre-image $f^{-1}(U)$ is $\mathcal{B}$-measurable.

Problem 11.8. Let $(X, \mathcal{B})$ be a measurable space and let $f \colon X \to \mathbb{R}$ be a measurable function. Prove that for every interval $I \subset \mathbb{R}$, the pre-image $f^{-1}(I)$ is $\mathcal{B}$-measurable.

Note: This is of course true for *open* intervals, since they are open sets. The point is then to show that it must also be true for closed or semi-closed intervals.

12. Twelfth week homework assignment

Let $(X, \mathcal{B}, \mu)$ be a measure space.

Problem 12.1. Prove that if $f, g \colon X \to \mathbb{R}$ are two measurable functions, and if $f \leq g$ μ-a.e., then

$$\int_X f \, d\mu \leq \int_X g \, d\mu.$$

Note: Do this in full detail, by starting with simple functions, then considering non-negative functions, then using the splitting into positive and negative parts.

Problem 12.2. Prove Markov's inequality in an abstract measure space: if $f \colon X \to \mathbb{R}$ is measurable, then for every $\lambda > 0$ we have

$$\mu\left(\{x \in X \colon \left|f(x)\right| \geq \lambda\}\right) \leq \frac{1}{\lambda} \int_X \left|f\right| d\mu \,.$$

Problem 12.3. Let $f \in L^1(X, \mu)$. Use Markov's inequality to show that all level sets $\{x \in X \colon \left|f(x)\right| \geq \lambda\}$ have finite measure. Then conclude that $\left|f\right| < \infty$ μ-a.e.

Problem 12.4. Prove that if $(X, \mathcal{B}, \mu)$ is a *finite* measure space, meaning that $\mu(X) < \infty$, and if $f_n \colon X \to [0, \infty]$, $n \geq 1$ is a sequence of measurable functions such that $f_n \to f$ *uniformly*, then

$$\int_X f_n \, d\mu \to \int_X f \, d\mu.$$

Problem 12.5. Let $0 \leq f_1 \leq f_2 \leq \ldots$ be an increasing sequence of measurable, non-negative functions on X. Let E be any $\mathcal{B}$-measurable set. Prove that

$$\lim_{n \to \infty} \int_E f_n \, d\mu = \int_E \left(\lim_{n \to \infty} f_n\right) d\mu \,.$$

Note: This is very similar to the statement of the monotone convergence theorem (MCT). The only difference is that the integration happens on a subset E and not on the whole space X. You need to find a way to apply the MCT.

Problem 12.6. Let $(X, \mathcal{B}, \mu)$ be a measure space and let $\phi\colon X \to [0,1]$ be a measurable function. For each $n \geq 1$ define the functions $f_n := \phi^{1/n}$ and $g_n := \phi^n$.

(a) Prove that the functions f_n and g_n are all measurable.
(b) Prove that

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \mu\left(\{x \in X \colon \phi(x) \neq 0\}\right).$$

(c) Prove that

$$\lim_{n\to\infty} \int_X g_n \, d\mu = \mu\left(\{x \in X \colon \phi(x) = 1\}\right).$$

Hint: In part (a) use the fact that power functions are continuous. In parts (b) and (c) use the MCT. Keep in mind that $0 \leq f(x) \leq 1$, which should tell you that the sequences f_n and g_n are monotonic.

Problem 12.7. Let $(X, \mathcal{B}, \mu)$ be a measure space and let $\phi\colon X \to [0, \infty]$ be a measurable function. Define the map $\nu\colon \mathcal{B} \to [0, \infty]$ as

$$\nu(A) := \int_A \phi \, d\mu = \int_X \phi \cdot \mathbf{1}_A \, d\mu \quad \text{for every } A \in \mathcal{B}.$$

(a) Prove that ν is a measure on $(X, \mathcal{B})$.
(b) Prove that if $f\colon X \to [0, \infty]$ is a measurable function, then

$$\int_X f \, d\nu = \int_X f \cdot \phi \, d\mu.$$

Hint: For part (a), just to warm up, begin by proving *finite* additivity of the map ν. To prove countable additivity, use Tonelli's theorem (the first consequence of the MCT).

Part (b) will be done in steps, starting from the simplest type of function f, to the most general.

i. Assume that f is an indicator function: $f = \mathbf{1}_E$ for some $E \in \mathcal{B}$.
ii. Assume that f is a simple function: $f = c_1 \mathbf{1}_{E_1} + \ldots + c_k \mathbf{1}_{E_k}$.
iii. For the general case, f measurable, use the fact that there is an increasing sequence $f_n \nearrow f$ of non-negative simple functions. Apply step **ii.** to f_n and then the MCT.

13. Thirteenth week homework assignment

Let $(X, \mathcal{B}, \mu)$ be a measure space.

Problem 13.1. Let $f_1, f_2, \dots : X \to \mathbb{R}$ be a sequence of measurable functions and assume that $f_n \to f$ in the L^1 norm as $n \to \infty$.

Prove that $f_n \to f$ in measure.

Hint: Use Markov's inequality.

Problem 13.2. Let $f_1, f_2, \dots : X \to \mathbb{R}$ be a sequence of measurable functions and assume that $f_n \to f$ almost uniformly as $n \to \infty$.

Prove that $f_n \to f$ in measure.

Problem 13.3. Assume that $\mu(X) < \infty$.

Let $f_1, f_2, \dots : X \to \mathbb{R}$ be a sequence of measurable functions and assume that $f_n \to f$ in the L^∞ norm as $n \to \infty$.

Prove that $f_n \to f$ in the L^1 norm.

Problem 13.4. Assume that $\mu(X) < \infty$.

Let $f_1, f_2, \dots : X \to \mathbb{R}$ be a sequence of measurable functions and assume that $f_n \to f$ pointwise a.e. as $n \to \infty$.

Prove that $f_n \to f$ in measure.

Hint: You may assume (why?) that the pointwise convergence is everywhere, instead of a.e. Define the sets

$$A_N := \{x \in X : \left|f_n(x) - f(x)\right| \le \epsilon \quad \text{for all } n \ge N\}.$$

Verify that $A_N \nearrow X$, so ${A_N}^{\complement} \searrow \emptyset$.

Apply the downwards monotone convergence theorem for sets ...

Problem 13.5. Prove the differentiation theorem for continuous functions with compact support. More precisely, prove the following: if $f : [a, b] \to \mathbb{R}$ is a continuous function, then for every $x \in (a, b)$ we have

$$\lim_{h \to 0^+} \frac{1}{h} \int_{[x, x+h]} f(t)\, dt = f(x).$$

Hint: You could derive this using the fundamental theorem of calculus I. You could also derive it more directly (and I would prefer that) simply by using the fact that a continuous function on a compact set is uniformly continuous.

Problem 13.6. (a) Compute the limit

$$\lim_{n\to\infty} \int_0^1 e^{-x^n}\, dx.$$

(b) Show that the function $g\colon [1, \infty) \to \mathbb{R}$, $g(x) := e^{-x}$ is absolutely integrable. Then compute the limit

$$\lim_{n\to\infty} \int_1^\infty e^{-x^n}\, dx.$$

Hint: In part (a) use the monotone convergence theorem. In part (b) use the dominated convergence theorem.

Problem 13.7. Show that the function $g\colon \mathbb{R} \to \mathbb{R}$ defined by $g(x) := e^{-x^2}$ is absolutely integrable (relative to the Lebesgue measure).

Problem 13.8. Define the sequence of functions $f_n\colon \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) := e^{-x^2} (\sin x)^n.$$

Show the following:

(a) $f_n \to 0$ pointwise a.e. (relative to the Lebesgue measure).

(b) $\lim_{n\to\infty} \int_{\mathbb{R}} f_n(x) dx = 0.$

(c) Prove also that $f_n \to 0$ in the L^1 norm (so in measure too).

Hint: It is helpful to remember that $\left|\sin x\right| \le 1$ for all x, and also that $\sin x = \pm 1$ does not occur often.

For part (b) use the dominated convergence theorem and the previous problem. In part (c) you have almost nothing left to do, if you realize that parts (a) and (b) are still true with $\left|f_n(x)\right|$ instead of $f_n(x)$.