REVIEW PROBLEMS
TMA4225-FOUNDATIONS OF ANALYSIS-FALL 2015

Problem 1. Let $p(x)$ be a polynomial function. Prove that the set

$$\{x \in [0,1] \colon \, \big|p(x)\big| \le 0\}$$

is *Jordan* measurable.

Hint: Consider the boundary of this set.

Is the same true for continuous functions? How about Lebesgue measurability?

Problem 2. Let $(X, \mathcal{B}, \mu)$ be a measure space, let $f \colon X \to \mathbb{R}$ be an absolutely integrable function and let M and m be two real numbers.

a) Prove that if $f(x) \le M$ for a.e. $x \in X$, then

$$\int_X f \, d\mu \le M \cdot \mu(X).$$

b) Prove that if $f(x) \ge m$ for a.e. $x \in X$, then

$$\int_X f \, d\mu \ge m \cdot \mu(X).$$

Problem 3. Let $f \colon [a,b] \to \mathbb{R}$ be a continuous function.

Prove that f is Lebesgue absolutely integrable and that

$$\int_a^b f(x) dx = \int_{[a,b]} f(x) dx \,,$$

where the left hand side refers to the Riemann-Darboux integral of f, while the right hand side to the Lebesgue integral of f.

To make it easier, divide the problem into the following steps:

a) Explain why f "lives in a box".

b) Prove that $\int_a^b f(x)dx \le \int_{[a,b]} f(x)dx$ by using approximations by step / simple functions from below.

c) Prove that $\int_a^b f(x)dx \ge \int_{[a,b]} f(x)dx$ by using approximations by step / simple functions from above.

Problem 4. Compute the following limit

$$\lim_{n\to\infty} \int_0^1 n\, x^2 \sin\left(\frac{1}{nx}\right) dx\,.$$

Hint: It helps to remember that

$$\lim_{t\to 0} \frac{\sin t}{t} = 1\,.$$

Problem 5. Compute the limit

$$\lim_{n\to\infty} \int_0^1 \sin\left(\frac{x}{n}\right) \cdot \ln x \, dx\,.$$

To make it easier, we may divide the problem into:

a) Prove that the function $g(x) := \ln x$ is absolutely integrable on $[0, 1]$.
b) For every $n \ge 1$ consider the function $f_n(x) := \sin\left(\frac{x}{n}\right) \cdot \ln x$. Explain why these functions are Lebesgue measurable on $[0, 1]$.
c) Compute the pointwise limit of this sequence of functions and then apply the appropriate convergence theorem.

Problem 6. Let $f \colon \mathbb{R} \to \mathbb{R}$ be an absolutely integrable function, let a be a fixed real number and define the function

$$F(x) := \int_{[a,x]} f(t)\, dt\,.$$

Prove that F is continuous everywhere.

Hint: Use the dominated convergence theorem.

Problem 7. Let $(X, \mathcal{B}, \mu)$ be a measure space and let $f_1,\ f_2,\ \ldots : X \to [0, \infty]$ be a sequence of measurable functions. Prove that if

$$\sum_{n=1}^{\infty} \int_X f_n \, d\mu < \infty\,,$$

then

$$\lim_{n\to\infty} f_n(x) = 0 \quad \text{for } \mu \text{ a.e. } x \in X.$$

Hint: Use Tonelli's convergence theorem.

Problem 8. Let $(X, \mathcal{B}, \mu)$ be a measure space and let $f \colon X \to \mathbb{R}$ be an absolutely integrable function. For every $n \geq 1$ define the set

$$E_n := \{x \in X \colon \big|f(x)\big| \geq n^2\}\,.$$

Prove that

$$\sum_{n=1}^{\infty} \mu(E_n) < \infty.$$

Hint: Use Markov's inequality.

Problem 9. Let $f \colon \mathbb{R} \to \mathbb{R}$ be an absolutely integrable function. Prove that for all $t \in \mathbb{R}$ we have

$$\int_{\mathbb{R}} f(x+t)\, dx = \int_{\mathbb{R}} f(x)\, dx\,.$$

Hint: Consider first the case when $f = \mathbf{1}_E$ is an indicator function (you need to apply the translation invariance of the Lebesgue measure, homework problem 5.1). Then let f be a simple function. After that let f be non-negative and approximate it by simple functions. Finally, split f into its positive and negative parts.

Problem 10. Let $f \colon \mathbb{R} \to [0, \infty)$ be a measurable function and let

$$A := \{(x, y) \in \mathbb{R}^2 \colon x \in \mathbb{R} \;\text{ and } 0 < y < f(x)\}$$

be the region under the graph of f.

Prove that A is a Lebesgue measurable set in $\mathbb{R}^2$ and that its measure is the Lebesgue integral of f, that is

$$m(A) = \int_{\mathbb{R}} f(x)\, dx.$$

Hint: First verify these statements when f is an indicator function: $f = \mathbf{1}_E$, with E Lebesgue measurable in $\mathbb{R}$. Then do it for a simple function. Finally, for the general case, consider an approximation by an increasing sequence of simple functions and apply the monotone convergence theorem.

Problem 11. Let $(X, \mathcal{B})$ be a measurable space. Let μ_1 and μ_2 be two measures on $(X, \mathcal{B})$ and define $\mu := \mu_1 + \mu_2$. More precisely, for every $A \in \mathcal{B}$, let

$$\mu(A) := \mu_1(A) + \mu_2(A)\,.$$

a) Prove that μ is a measure on $(X, \mathcal{B})$.
b) Prove that for every measurable function $f \colon X \to [0, \infty]$,

$$\int_X f\, d\mu = \int_X f\, d\mu_1 + \int_X f\, d\mu_2\,.$$