

REVIEW PROBLEMS
TMA4225-FOUNDATIONS OF ANALYSIS-FALL 2015

Problem 1. Let $p(x)$ be a polynomial function. Prove that the set

$$\{x \in [0, 1]: p(x) \leq 0\}$$

is *Jordan* measurable.

Hint: Consider the boundary of this set.

Is the same true for continuous functions? How about Lebesgue measurability?

Problem 2. Let (X, \mathcal{B}, μ) be a measure space, let $f: X \rightarrow \mathbb{R}$ be an absolutely integrable function and let M and m be two real numbers.

a) Prove that if $f(x) \leq M$ for a.e. $x \in X$, then

$$\int_X f d\mu \leq M \cdot \mu(X).$$

b) Prove that if $f(x) \geq m$ for a.e. $x \in X$, then

$$\int_X f d\mu \geq m \cdot \mu(X).$$

Problem 3. Let $f: [a, b] \rightarrow \mathbb{R}$ be a continuous function.

Prove that f is Lebesgue absolutely integrable and that

$$\int_a^b f(x)dx = \int_{[a,b]} f(x)d\mu,$$

where the left hand side refers to the Riemann-Darboux integral of f , while the right hand side to the Lebesgue integral of f .

To make it easier, divide the problem into the following steps:

- a) Explain why f “lives in a box”.
- b) Prove that $\int_a^b f(x)dx \leq \int_{[a,b]} f(x)d\mu$ by using approximations by step / simple functions from below.
- c) Prove that $\int_a^b f(x)dx \geq \int_{[a,b]} f(x)d\mu$ by using approximations by step / simple functions from above.

Problem 4. Compute the following limit

$$\lim_{n \rightarrow \infty} \int_0^1 n x^2 \sin\left(\frac{1}{nx}\right) dx.$$

Hint: It helps to remember that

$$\lim_{t \rightarrow 0} \frac{\sin t}{t} = 1.$$

Problem 5. Compute the limit

$$\lim_{n \rightarrow \infty} \int_0^1 \sin\left(\frac{x}{n}\right) \cdot \ln x dx.$$

To make it easier, we may divide the problem into:

- Prove that the function $g(x) := \ln x$ is absolutely integrable on $[0, 1]$.
- For every $n \geq 1$ consider the function $f_n(x) := \sin\left(\frac{x}{n}\right) \cdot \ln x$. Explain why these functions are Lebesgue measurable on $[0, 1]$.
- Compute the pointwise limit of this sequence of functions and then apply the appropriate convergence theorem.

Problem 6. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be an absolutely integrable function, let a be a fixed real number and define the function

$$F(x) := \int_{[a, x]} f(t) dt.$$

Prove that F is continuous everywhere.

Hint: Use the dominated convergence theorem.

Problem 7. Let (X, \mathcal{B}, μ) be a measure space and let $f_1, f_2, \dots : X \rightarrow [0, \infty]$ be a sequence of measurable functions. Prove that if

$$\sum_{n=1}^{\infty} \int_X f_n d\mu < \infty,$$

then

$$\lim_{n \rightarrow \infty} f_n(x) = 0 \quad \text{for } \mu \text{ a.e. } x \in X.$$

Hint: Use Tonelli's convergence theorem.

Problem 8. Let (X, \mathcal{B}, μ) be a measure space and let $f: X \rightarrow \mathbb{R}$ be an absolutely integrable function. For every $n \geq 1$ define the set

$$E_n := \{x \in X : |f(x)| \geq n^2\}.$$

Prove that

$$\sum_{n=1}^{\infty} \mu(E_n) < \infty.$$

Hint: Use Markov's inequality.

Problem 9. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be an absolutely integrable function. Prove that for all $t \in \mathbb{R}$ we have

$$\int_{\mathbb{R}} f(x+t) dx = \int_{\mathbb{R}} f(x) dx.$$

Hint: Consider first the case when $f = \mathbf{1}_E$ is an indicator function (you need to apply the translation invariance of the Lebesgue measure, homework problem 5.1). Then let f be a simple function. After that let f be non-negative and approximate it by simple functions. Finally, split f into its positive and negative parts.

Problem 10. Let $f: \mathbb{R} \rightarrow [0, \infty)$ be a measurable function and let

$$A := \{(x, y) \in \mathbb{R}^2 : x \in \mathbb{R} \text{ and } 0 < y < f(x)\}$$

be the region under the graph of f .

Prove that A is a Lebesgue measurable set in \mathbb{R}^2 and that its measure is the Lebesgue integral of f , that is

$$m(A) = \int_{\mathbb{R}} f(x) dx.$$

Hint: First verify these statements when f is an indicator function: $f = \mathbf{1}_E$, with E Lebesgue measurable in \mathbb{R} . Then do it for a simple function. Finally, for the general case, consider an approximation by an increasing sequence of simple functions and apply the monotone convergence theorem.

Problem 11. Let (X, \mathcal{B}) be a measurable space. Let μ_1 and μ_2 be two measures on (X, \mathcal{B}) and define $\mu := \mu_1 + \mu_2$. More precisely, for every $A \in \mathcal{B}$, let

$$\mu(A) := \mu_1(A) + \mu_2(A).$$

- a) Prove that μ is a measure on (X, \mathcal{B}) .
- b) Prove that for every measurable function $f: X \rightarrow [0, \infty]$,

$$\int_X f d\mu = \int_X f d\mu_1 + \int_X f d\mu_2.$$