The definition of the Jordan measure

Let $E \subset \mathbb{R}^d$ be a bounded set.

- The Jordan inner measure of E:

$$m_{*,J}(E) := \sup\{m(A)\colon A \subset E,\ A \text{ elementary set}\}.$$

- The Jordan outer measure of E:

$$m^{*,J}(E) := \inf\{m(B)\colon E \subset B,\ B \text{ elementary set}\}.$$

Note that $m_{*,J}(E) \leq m^{*,J}(E)$ for all bounded sets E.

If $m_{*,J}(E) = m^{*,J}(E)$ then we say that E is *Jordan measurable*. We denote by $m(E)$ the common value of the inner and the outer Jordan measures and we call it the *Jordan measure* of E.

We denote by $\mathcal{J}(\mathbb{R}^d)$ the family of all Jordan measurable sets in $\mathbb{R}^d$.

Clearly if E is elementary then it is also Jordan measurable. In other words, $\mathcal{E}(\mathbb{R}^d) \subset \mathcal{J}(\mathbb{R}^d)$.

A more helpful characterization of Jordan measurability

Let $E \subset \mathbb{R}^d$ be a bounded set. The definition of Jordan measurability is natural but not easy to verify directly.

Firstly, note the following: if the outer Jordan measure of E is 0, that is if $m^{*,J}(E) = 0$, then E is Jordan measurable and $m(E) = 0$. This is sometimes useful, for instance when showing that Cantor's set is Jordan measurable and of Jordan measure zero.

The following lemma can be seen as a more practical reformulation of the definition of Jordan measurability.

Lemma: The following are equivalent (abbreviated TFAE):

- E is Jordan measurable.
- For every $\epsilon > 0$ there are elementary sets A and B with $A \subset E \subset B$ such that $m(B \setminus A) < \epsilon$.
- For every $\epsilon > 0$ there is an elementary set A such that $m^{*,J}(A \triangle E) < \epsilon$.

Proposition (properties of Jordan measurable sets):

If E and F are Jordan measurable sets, then

$$E \cup F, \; E \cap F, \; E \setminus F, \; E \triangle F$$

are Jordan measurable as well.

The Jordan measure satisfies the following properties:

(1) (non-negative) $m(E) \geq 0$ for all $E \in \mathcal{J}(\mathbb{R}^d)$.

(2) (finitely-additive) $m(E \cup F) = m(E) + m(F)$ for all $E, F \in \mathcal{J}(\mathbb{R}^d)$ with $E \cap F = \emptyset$.

(3) (translation invariant) $m(E + x) = m(E)$ for all $E \in \mathcal{J}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$.

Moreover,

(4) (monotone) If $E, F \in \mathcal{J}(\mathbb{R}^d)$ with $E \subset F$, then $m(E) \leq m(F)$.

(5) (sub-additive) $m(E \cup F) \leq m(E) + m(F)$ for all $E, F \in \mathcal{J}(\mathbb{R}^d)$.