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Examination paper for
TMA4225 Foundations of Analysis
Solutions

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You were required to provide detailed explanations for each problem. For pedagogical reasons, my solutions will be even more detailed than (most of) yours.

Problem 1

a) Define the concept of Lebesgue measurable sets in $\mathbb{R}^d$.

Solution. In class we took as definition of measurability what is commonly known as "Littlewood's first principle": a set $E \subset \mathbb{R}^d$ is Lebesgue measurable if it is "almost open". That is, E is Lebesgue measurable if for every $\epsilon > 0$ there is an open set $U \supset E$ such that $m^*(U \setminus E) \leq \epsilon$.

Another way to define Lebesgue measurable sets in $\mathbb{R}^d$ is via "Carathéodory's criterion": a set $E \subset \mathbb{R}^d$ is Lebesgue measurable if for every box $B \subset \mathbb{R}^d$ we have

$$|B| = m^*(B \cap E) + m^*(B \setminus E)\,,$$

where $|B|$ refers to the volume of the box B.

These two definitions were shown to be equivalent, so you could have used either one of them as your answer. □

b) Prove that if $E \subset \mathbb{R}^d$ is a Lebesgue measurable set, then for every $\epsilon > 0$ there are a closed set $F \subset E$ and an open set $U \supset E$ such that

$$m(U) \leq m(F) + \epsilon.$$

You may use (without proof) any properties of the Lebesgue measurable sets discussed in class.

Solution. Let $E \subset \mathbb{R}^d$ be a Lebesgue measurable set and let $\epsilon > 0$.

By the definition of Lebesgue measurability, there is an open set $U \supset E$ such that

$$m^*(U \setminus E) \leq \frac{\epsilon}{2}\,. \tag{1b-i}$$

By the "inner approximation by closed sets" property of Lebesgue measurable sets, there is a closed set $F \subset E$ such that

$$m^*(E \setminus F) \leq \frac{\epsilon}{2}\,. \tag{1b-ii}$$

We have $F \subset E \subset U$, so

$$U \setminus F = (U \setminus E) \cup (E \setminus F)\,, \tag{1b-iii}$$

and since open and closed sets are Lebesgue measurable, U, F and then $U \setminus F$ are Lebesgue measurable.

Combining (1b-i), (1b-ii) and (1b-iii) we obtain

$$m(U \setminus F) = m^*(U \setminus F) \leq m^*(U \setminus E) + m^*(E \setminus F) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\,.$$

To conclude, note that since $F \subset U$, by the additivity of the Lebesgue measure we have

$$m(U) = m(F) + m(U \setminus F) \leq m(F) + \epsilon\,.$$

□

Problem 2 This problem is about the concept of *Lebesgue measurable functions.*

a) Let $f\colon \mathbb{R}^d \to [0, +\infty]$ be a function. Write the definition of Lebesgue measurability for such a function.

Solution. A function $f\colon \mathbb{R}^d \to [0, +\infty]$ is called Lebesgue measurable if it is the pointwise limit of a sequence of non-negative *simple* functions, in other words, if there is a sequence $f_1, f_2, \ldots : \mathbb{R}^d \to [0, +\infty]$ of *simple* functions such that

$$f_n(x) \to f(x) \quad \text{as } n \to \infty \quad \text{for every } x \in \mathbb{R}^d\,.$$

□

b) Provide at least *four* other equivalent formulations of Lebesgue measurability for a nonnegative function. No proofs are required.

Solution. The following statements are equivalent for a function $f\colon \mathbb{R}^d \to [0, +\infty]$.

(0) f is Lebesgue measurable.

(1) There is a sequence $f_1, f_2, \ldots : \mathbb{R}^d \to [0, +\infty]$ of simple functions such that

$$f_n(x) \to f(x) \quad \text{as } n \to \infty \quad \text{for } \textit{almost} \text{ every } x \in \mathbb{R}^d .$$

(2) There is an *increasing* sequence $0 \le f_1 \le f_2 \le \ldots$ of simple functions such that each f_n "lives in a box" (i.e. it is bounded and it has bounded support) and

$$f(x) = \sup_{n \ge 1} f_n(x) \quad \text{for all } x \in \mathbb{R}^d .$$

(3a) For every $\lambda \in [0, +\infty]$, the set $\{x \in \mathbb{R}^d : f(x) > \lambda\}$ is Lebesgue measurable.

(3b) For every $\lambda \in [0, +\infty]$, the set $\{x \in \mathbb{R}^d : f(x) \ge \lambda\}$ is Lebesgue measurable.

(3c) For every $\lambda \in [0, +\infty]$, the set $\{x \in \mathbb{R}^d : f(x) < \lambda\}$ is Lebesgue measurable.

(3d) For every $\lambda \in [0, +\infty]$, the set $\{x \in \mathbb{R}^d : f(x) \le \lambda\}$ is Lebesgue measurable.

(4) For every interval $I \subset [0, +\infty]$, the set $f^{-1}(I) = \{x \in \mathbb{R}^d : f(x) \in I\}$ is Lebesgue measurable.

(5) For every open set $U \subset [0, +\infty]$, the set $f^{-1}(U)$ is Lebesgue measurable.

(6) For every closed set $F \subset [0, +\infty]$, the set $f^{-1}(F)$ is Lebesgue measurable.

Note: Since all of these statements are equivalent, you could have chosen any one of them as the definition of Lebesgue measurability in part a), and any other four of them as your answer in part b).

□

c) Let $f : \mathbb{R}^d \to \mathbb{R}$. That is, consider a function that may take both positive and negative values. Define what it means for such a function to be Lebesgue measurable.

Solution. A function $f : \mathbb{R}^d \to \mathbb{R}$ is called Lebesgue measurable if it is the pointwise a.e. limit of a sequence $f_1, f_2, \ldots : \mathbb{R}^d \to \mathbb{R}$ of simple functions. □

Note: You may chose any other equivalent formulation as a definition.

d) Prove that $f : \mathbb{R}^d \to \mathbb{R}$ is Lebesgue measurable if and only if f^+ and f^- are both Lebesgue measurable.

Solution. We will simply use the definition of measurable functions stated above.

" $\implies$ ": Assume that $f\colon \mathbb{R}^d \to \mathbb{R}$ is Lebesgue measurable. Then there is a sequence $f_1, f_2, \ldots : \mathbb{R}^d \to \mathbb{R}$ of simple functions such that

$$f_n(x) \to f(x) \quad \text{for a.e. } x \in \mathbb{R}^d\,.$$

We know that if $\{y_n\}_{n\geq 1}$ is a sequence of real numbers such that $y_n \to y \in \mathbb{R}$, then $y_n^+ \to y^+$ and $y_n^- \to y^-$. Hence for a.e. $x \in \mathbb{R}^d$ we have that

$$f_n^+(x) \to f^+(x) \quad \text{and} \quad f_n^-(x) \to f^-(x) \quad \text{as } n \to \infty\,.$$

Clearly f_n^+ and f_n^- are also simple functions, so f^+ and f^- are pointwise a.e. limits of simple functions, which proves that they are Lebesgue measurable.

" $\impliedby$ ": Assume that f^+ and f^- are Lebesgue measurable functions. Then each of them is a pointwise limit of a sequence of (nonnegative) simple functions. That is, there are sequences $\{g_n\}_{n\geq 1}$ and $\{h_n\}_{n\geq 1}$ of simple functions, which we may choose to be bounded (so also finite) such that for all $x \in \mathbb{R}^d$,

$$g_n(x) \to f^+(x) \quad \text{and} \quad h_n(x) \to f^-(x) \quad \text{as } n \to \infty\,.$$

For each $n \geq 1$, since g_n, $h_n\colon \mathbb{R}^d \to [0, \infty)$ are simple and finite, the difference $g_n - h_n\colon \mathbb{R}^d \to \mathbb{R}$ is a simple function as well. Moreover, for all $x \in \mathbb{R}^d$ we have

$$(g_n - h_n)(x) = g_n(x) - h_n(x) \to f^+(x) - f^-(x) = f(x) \quad \text{as } n \to \infty\,,$$

showing that f is the pointwise limit of the sequence $\{(g_n - h_n)\}_{n\geq 1}$ of simple function. Therefore, f is Lebesgue measurable.

Note: Another way to solve this problem, which some of you chose, would be using the "measurable level sets" characterization of Lebesgue measurable functions. □

e) Prove that if $f\colon \mathbb{R}^d \to \mathbb{R}$ is continuous everywhere, then f is Lebesgue measurable.

Solution. If $f\colon \mathbb{R}^d \to \mathbb{R}$ is a continuous function, then for every open set $U \subset \mathbb{R}$ we have $f^{-1}(U)$ is open, hence also Lebesgue measurable. This proves that f is a measurable function. □

f) Prove that if $f\colon \mathbb{R}^d \to \mathbb{R}$ is continuous *almost everywhere*, then f is Lebesgue measurable.

Solution. We use the characterization (i.e. equivalent statement) of Lebesgue measurable functions in terms of level sets. More precisely, we know that a function $f\colon \mathbb{R}^d \to \mathbb{R}$ is Lebesgue measurable if and only if for every $\lambda \in \mathbb{R}$, the level set

$$\{x \in \mathbb{R}^d \colon f(x) > \lambda\}$$

is Lebesgue measurable in $\mathbb{R}^d$.

Fix $\lambda \in \mathbb{R}$ and denote the corresponding level set by

$$E := \{x \in \mathbb{R}^d \colon f(x) > \lambda\} .$$

Let $x \in E$. Then

i. Either f is *not* continuous at x.
Let $\mathcal{N}$ be the set of all such points. Since f is continuous almost everywhere, it follows that $\mathcal{N}$ has zero Lebesgue measure, and in particular it is Lebesgue measurable (here we are using the fact that the Lebesgue measure is complete).

ii. Or f is continuous at x.
In this case, as $f(x) > \lambda$, by continuity, if y is close enough to x, then $f(y) > \lambda$. Indeed, let $\epsilon := f(x) - \lambda > 0$. There is $\delta > 0$ small enough so that if $|y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$, which then implies $f(y) > \lambda$, so $y \in E$.
Therefore, in this case we obtained an open interval $I_x := (x - \delta, x + \delta)$ centered at x such that $I_x \subset E$.

We conclude that

$$E = \mathcal{N} \cup \bigcup_{x \text{ as in case ii.}} I_x .$$

A union of open intervals (namely the intervals I_x) is an open set, hence also a Lebesgue measurable set. Sine $\mathcal{N}$ is measurable, we conclude that E is Lebesgue measurable, which completes the proof. □

Note: Some of you used a more direct argument for case ii. You said that the restriction $f\big|_{\mathbb{R}^d \setminus \mathcal{N}}$ was continuous, thus the pre-image of an open set $U \subset \mathbb{R}$ must be *open* and so Lebesgue measurable as well. I agree with this argument, but it needs a clarification.

The pre-image of U under $f\big|_{\mathbb{R}^d \setminus \mathcal{N}}$ is indeed open, but in the topological space $\mathbb{R}^d \setminus \mathcal{N}$, not in $\mathbb{R}^d$. Which is fine, since this means that it is the intersection of an open (hence measurable) set $\mathcal{D}$ in $\mathbb{R}^d$ with $\mathbb{R}^d \setminus \mathcal{N}$ (which is also measurable, since $\mathcal{N}$ is measurable). Thus our pre-image is the intersection of two measurable sets, so it is measurable as well.

Problem 3 Compute the limit

$$\lim_{n\to\infty} \int_0^\infty \frac{n\,\sqrt{x}}{1+n^2\,x^2}\,dx\,.$$

To make it easier, you can divide up the problem into the following steps.

a) Show that the function $\dfrac{1}{x^{3/2}}$ is absolutely integrable on $[1,\infty)$ and that the function $\dfrac{1}{2\sqrt{x}}$ is absolutely integrable on $[0,1]$.

Solution. The function $\dfrac{1}{x^{3/2}}$ is continuous on $[1,\infty)$, hence it is Lebesgue measurable. It is also non-negative. Thus to show that it is absolutely integrable, we only need to show that its Lebesgue integral is finite. By the vertical truncation property (or by the monotone convergence theorem), we have that

$$\int_{[1,\infty)} \frac{1}{x^{3/2}}\,dx = \lim_{n\to\infty} \int_{[1,n)} \frac{1}{x^{3/2}}\,dx\,. \tag{3a-i}$$

Since continuous functions on a bounded interval are both Riemann and Lebsegue integrable and the values of these two integrals are the same, we have

$$\int_{[1,n)} \frac{1}{x^{3/2}}\,dx = \int_1^n \frac{1}{x^{3/2}}\,dx = -2x^{-1/2}\Big|_1^n = -\frac{2}{n^{1/2}} + 2\,.$$

This combined with (3a-i) implies that

$$\int_{[1,\infty)} \frac{1}{x^{3/2}}\,dx = \lim_{n\to\infty} \Big(-\frac{2}{n^{1/2}} + 2\Big) = 2 < \infty\,,$$

showing that the function $\dfrac{1}{x^{3/2}}$ is absolutely integrable on $[1,\infty)$.

The function $\dfrac{1}{2\sqrt{x}}$ is continuous at every point in $[0,1]$ except at 0, hence it is Lebesgue measurable on $[0,1]$. It is also non-negative. Thus to show that it is absolutely integrable, we only need to show that its Lebesgue integral is finite. By the monotone convergence theorem we have that

$$\int_{[0,1]} \frac{1}{2\sqrt{x}}\, dx = \lim_{\epsilon \to 0^+} \int_{[\epsilon,1]} \frac{1}{2\sqrt{x}}\, dx\,. \tag{3a-ii}$$

Again, since continuous functions on a bounded interval are both Riemann and Lebesgue integrable and the values of these two integrals are the same, we have

$$\int_{[\epsilon,1]} \frac{1}{2\sqrt{x}}\, dx = \int_\epsilon^1 \frac{1}{2\sqrt{x}}\, dx = x^{1/2}\Big|_\epsilon^1 = 1 - \epsilon^{1/2}\,.$$

This combined with (3a-ii) implies that

$$\int_{[0,1]} \frac{1}{2\sqrt{x}}\, dx = \lim_{\epsilon \to 0^+} (1 - \epsilon^{1/2}) = 1 < \infty\,,$$

showing that the function $\frac{1}{2\sqrt{x}}$ is absolutely integrable on $[0, 1]$. □

b) For every $n \geq 1$ define the function $f_n(x) := \frac{n\sqrt{x}}{1 + n^2 x^2}$. Explain why f_n is Lebesgue measurable on $[0, \infty)$.

Solution. The function $f_n(x) := \frac{n\sqrt{x}}{1 + n^2 x^2}$ is continuous everywhere on $[0, \infty)$, hence it is Lebesgue measurable on $[0, \infty)$, and in fact on any other measurable subset of this interval.

Let us also note, for the next two questions, that for every $x \in [0, \infty)$ we have

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{n\sqrt{x}}{1 + n^2 x^2} = 0\,. \tag{3-limit}$$

□

c) Compute the limit

$$\lim_{n\to\infty} \int_1^\infty \frac{n\sqrt{x}}{1 + n^2 x^2}\, dx\,.$$

Solution. We use the dominated convergence theorem. As noted in part b), the functions $f_n(x)$ are Lebesgue measurable on $[1, \infty)$. Moreover, for $n \geq 1$ and for all $x \in [1, \infty)$,

$$\left|f_n(x)\right| = \frac{n\sqrt{x}}{1 + n^2 x^2} \leq \frac{n\sqrt{x}}{n^2 x^2} = \frac{1}{n}\,\frac{1}{x^{3/2}} \leq \frac{1}{x^{3/2}}\,.$$

In other words, the sequence $f_n(x)$ is dominated by the function $\frac{1}{x^{3/2}}$, which was shown to be absolutely integrable on the interval $[1, \infty)$.

Therefore, the dominated convergence theorem is applicable and we have

$$\lim_{n\to\infty} \int_1^\infty \frac{n\,\sqrt{x}}{1+n^2\,x^2}\,dx = \lim_{n\to\infty} \int_{[1,\infty)} \frac{n\,\sqrt{x}}{1+n^2\,x^2}\,dx$$
$$= \int_{[1,\infty)} \Big[\lim_{n\to\infty} \frac{n\,\sqrt{x}}{1+n^2\,x^2}\Big]\,dx = \int_{[1,\infty)} 0\,dx = 0\,.$$

□

d) Compute the limit

$$\lim_{n\to\infty} \int_0^1 \frac{n\,\sqrt{x}}{1+n^2\,x^2}\,dx\,.$$

Solution. Again, we use the dominated convergence theorem. As noted in part b), the functions $f_n(x)$ are Lebesgue measurable on $[0, 1]$. Moreover, for $n \geq 1$ and for all $x > 0$, we have that

$$\frac{n\,\sqrt{x}}{1+n^2\,x^2} \leq \frac{1}{2\sqrt{x}}\,.$$

Indeed, this is equivalent with

$$n\,\sqrt{x}\cdot 2\sqrt{x} \leq 1 + n^2x^2$$

or

$$2nx \leq 1 + n^2x^2$$

or

$$0 \leq 1 - 2nx + n^2x^2$$

or

$$0 \leq (1 - nx)^2\,,$$

which is of course true.

Therefore, for a.e. $x \in [0, 1]$ (more precisely, with the exception of $x = 0$) we have that the sequence $f_n(x)$ is dominated by the function $\frac{1}{2\sqrt{x}}$, which was shown to be absolutely integrable on $[0, 1]$. The dominated convergence theorem is then applicable and we have

$$\lim_{n\to\infty} \int_0^1 \frac{n\,\sqrt{x}}{1+n^2\,x^2}\,dx = \lim_{n\to\infty} \int_{[0,1]} \frac{n\,\sqrt{x}}{1+n^2\,x^2}\,dx$$
$$= \int_{[0,1]} \Big[\lim_{n\to\infty} \frac{n\,\sqrt{x}}{1+n^2\,x^2}\Big]\,dx = \int_{[0,1]} 0\,dx = 0\,.$$

□

e) Combine your answers from parts c) and d) to draw the conclusion.

Solution. We have

$$\int_0^\infty \frac{n\,\sqrt{x}}{1+n^2\,x^2}\,dx = \int_0^1 \frac{n\,\sqrt{x}}{1+n^2\,x^2}\,dx + \int_1^\infty \frac{n\,\sqrt{x}}{1+n^2\,x^2}\,dx\,.$$

Then combining the answers in parts c) and d) we conclude that

$$\lim_{n\to\infty} \int_0^\infty \frac{n\,\sqrt{x}}{1+n^2\,x^2}\,dx = 0+0=0\,.$$

□

Problem 4 Prove that convergence in L^1 norm implies convergence in measure. More precisely, let $(X, \mathcal{B}, \mu)$ be a measure space, let $f\colon X \to \mathbb{R}$ be a measurable function and let $f_1, f_2, \ldots : X \to \mathbb{R}$ be a sequence of measurable functions.

a) Define the following concept: $f_n \to f$ in the L^1 norm.

Solution. The sequence $f_n \to f$ in the L^1 norm if

$$\|f_n - f\|_{L^1} \to 0 \quad \text{as } n \to \infty\,,$$

or, in other words, if

$$\lim_{n\to\infty} \int_X \left|f_n - f\right|\,d\mu = 0\,.$$

□

b) Define the following concept: $f_n \to f$ in measure.

Solution. The sequence $f_n \to f$ in measure if for every $\epsilon > 0$ we have

$$\mu\left(\{x \in X\colon\ |f_n(x) - f(x)| \geq \epsilon\}\right) \to 0 \quad \text{as } n \to \infty\,.$$

□

c) Prove that if $f_n \to f$ in the L^1 norm then $f_n \to f$ in measure.

Solution. Assume that $f_n \to f$ in the L^1 norm, so

$$\int_X |f_n - f| \, d\mu \to 0 \quad \text{as} \quad n \to \infty .$$

Fix $\epsilon > 0$. Applying Markov's inequality to the measurable, nonnegative function $g_n(x) := |f_n(x) - f(x)|$, we have

$$\mu\left(\{x \in X \colon g(x) \geq \epsilon\}\right) \leq \frac{\int_X g \, d\mu}{\epsilon} ,$$

so

$$\mu\left(\{x \in X \colon |f_n(x) - f(x)| \geq \epsilon\}\right) \leq \frac{\int_X |f_n - f| \, d\mu}{\epsilon} \to \frac{0}{\epsilon} = 0 ,$$

proving the convergence in measure of f_n to f. □

Problem 5 Let $(X, \mathcal{B})$ be a measurable space, let $x_0 \in X$ and denote by δ_{x_0} the Dirac measure centered at x_0. More precisely, for any $A \in \mathcal{B}$,

$$\delta_{x_0}(A) := \begin{cases} 1 & \text{if } x_0 \in A \\ 0 & \text{if } x_0 \notin A. \end{cases}$$

a) Prove that δ_{x_0} is indeed a measure on $(X, \mathcal{B})$.

Solution. We verify the two axioms of a measure.

i. Since $x_0 \notin \emptyset$ it follows that $\delta_{x_0}(\emptyset) = 0$.

ii. Let $\{E_n\}_{n \geq 1}$ be *disjoint* $\mathcal{B}$-measurable sets.

If $x_0 \notin \bigcup_{n \geq 1} E_n$, then $x_0 \notin E_n$ for every $n \geq 1$, so

$$\delta_{x_0}\Big(\bigcup_{n \geq 1} E_n\Big) = 0 \quad \text{and} \quad \delta_{x_0}(E_n) = 0 \quad \text{for all} \;\; n \geq 1 .$$

Thus

$$\delta_{x_0}\Big(\bigcup_{n \geq 1} E_n\Big) = 0 = \sum_{n=1}^{\infty} \delta_{x_0}(E_n) .$$

If $x_0 \in \bigcup_{n \geq 1} E_n$, since the sets E_n are disjoint, then x_0 belongs to *exactly one* such set, say to E_{n_0}, so

$$\delta_{x_0}\Big(\bigcup_{n \geq 1} E_n\Big) = 1, \quad \delta_{x_0}(E_{n_0}) = 1 \quad \text{and} \quad \delta_{x_0}(E_n) = 0 \quad \text{for all} \;\; n \neq n_0 .$$

Thus

$$\delta_{x_0}\Big(\bigcup_{n\geq 1} E_n\Big) = 1 = 1 + 0 + 0 + \ldots = \delta_{x_0}(E_{n_0}) + \sum_{n\neq n_0} \delta_{x_0}(E_n) = \sum_{n=1}^{\infty} \delta_{x_0}(E_n)\,.$$

In both case we have shown that $\delta_{x_0}\Big(\bigcup_{n\geq 1} E_n\Big) = \sum_{n=1}^{\infty} \delta_{x_0}(E_n)$, so δ_{x_0} is countably additive. □

b) Prove that for any measurable function $f\colon X \to \mathbb{R}$ we have

$$\int_X f \; d\,\delta_{x_0} = f(x_0)\,, \tag{5b}$$

that is, the integral of any function with respect to the Dirac measure centered at x_0 is simply the value of the function at the point x_0.

Solution. We first prove formula (5b) for any indicator function, then for any simple function, then for measurable, non-negative functions, and finally, for any measurable function $f\colon X \to \mathbb{R}$. Before we begin, we make the easy to verify observation that for any $\mathcal{B}$-measurable set E we have

$$\delta_{x_0}(E) = \mathbf{1}_E(x_0)\,.$$

i. Let $f = \mathbf{1}_E$, where E is any $\mathcal{B}$-measurable set. Then by the definition of the integral and the above observation,

$$\int_X \mathbf{1}_E \, d\delta_{x_0} = \delta_{x_0}(E) = \mathbf{1}_E(x_0)\,,$$

which proves (5b) for the function $\mathbf{1}_E$.

ii. Let $f = \sum_{i=1}^{k} c_i \mathbf{1}_{E_i}$ be any simple function. Using the linearity of the integral and step i. above, we have

$$\int_X \Big(\sum_{i=1}^{k} c_i \mathbf{1}_{E_i}\Big)\, d\delta_{x_0} = \sum_{i=1}^{k} c_i \int_X \mathbf{1}_{E_i}\, d\delta_{x_0} = \sum_{i=1}^{k} c_i \; \mathbf{1}_{E_i}(x_0) = \Big(\sum_{i=1}^{k} c_i \mathbf{1}_{E_i}\Big)(x_0)\,,$$

which establishes (5b) for simple functions.

iii. Let $f\colon X \to [0, +\infty]$ be a $\mathcal{B}$-measurable function.
Then there is an *increasing* sequence $0 \le f_1 \le f_2 \le \dots$ of *simple* functions such that $f_n(x) \to f(x)$ as $n \to \infty$ for *all* $x \in X$.

Since f_n are all simple functions, step ii. above applies to them, so for all $n \ge 1$ we have

$$\int_X f_n \, d\,\delta_{x_0} = f_n(x_0)\,. \tag{5b-iii}$$

We will take the limit as $n \to \infty$ on both sides of the formula (5b-iii).

Since the sequence $\{f_n\}_{n\ge1}$ is increasing, the monotone convergence theorem is applicable, so

$$\lim_{n\to\infty} \int_X f_n \, d\,\delta_{x_0} = \int_X \Big(\lim_{n\to\infty} f_n\Big)\, d\,\delta_{x_0} = \int_X f \, d\delta_{x_0}\,.$$

Moreover, since $f_n(x) \to f(x)$ for all $x \in X$, this also holds for $x = x_0$, so

$$\lim_{n\to\infty} f_n(x_0) = f(x_0)\,.$$

The last two calculations together with formula (5b-iii) imply

$$\int_X f \, d\,\delta_{x_0} = f(x_0)\,,$$

proving (5b) for measurable nonnegative functions.

Note: As some of you noted, instead of using the monotone convergence theorem, you could have treated this case using just the definition of the integral:

$$\int_X f \, d\delta_{x_0} = \sup\left\{ \int_X s \, d\delta_{x_0} \,:\, 0 \le s \le f,\ s \text{ is a simple function} \right\}.$$

Indeed, by step ii., (5b) applies to simple functions, so if s is simple then

$$\int_X s \, d\delta_{x_0} = s(x_0)\,.$$

Taking the supremum over all functions $0 \le s \le f$ that are simple and using the definition of the integral above, we get

$$\int_X f \, d\delta_{x_0} = \sup\,\{s(x_0) \,:\, 0 \le s \le f,\ s \text{ is a simple function}\,\} \le f(x_0)\,,$$

hence

$$\int_X f \, d\delta_{x_0} \leq f(x_0)\,.$$

To see that we actually have equality, note that the constant function $s(x) = f(x_0)$ for all $x \in X$ is obviously simple (because $s = f(x_0)\,\mathbf{1}_X$) and of course

$$\int_X s \, d\delta_{x_0} = s(x_0) = f(x_0)\,,$$

thus the supremum is attained for this constant function and it is $f(x_0)$.

iv. Finally let $f\colon X \to \mathbb{R}$ be any measurable function.
Then f^+ and f^- are also measurable but also non-negative and finite. Step iii. is then applicable to f^+ and f^- and we have

$$\int_X f^+ \, d\,\delta_{x_0} = f^+(x_0) < \infty \quad \text{and} \quad \int_X f^- \, d\,\delta_{x_0} = f^-(x_0) < \infty\,.$$

Then f is absolutely integrable and

$$\int_X f \, d\,\delta_{x_0} = \int_X f^+ \, d\,\delta_{x_0} - \int_X f^- \, d\,\delta_{x_0} = f^+(x_0) - f^-(x_0) = f(x_0)\,,$$

proving (5b) in all cases. □

c) Consider the measure space $(X, \mathcal{B}, \delta_{x_0})$. Let $f_1, f_2, \ldots\, : X \to \mathbb{R}$ be a sequence of measurable functions. In this context, what does $f_n \to f$ in measure mean? What does $f_n \to f$ in the L^1 norm mean?

Solution. Applying the definition of convergence in measure to our setting we have: for every $\epsilon > 0$,

$$\delta_{x_0}\Big(\{x \in X\colon\, |f_n(x) - f(x)| \geq \epsilon\}\Big) \to 0 \quad \text{as } \; n \to \infty\,.$$

The Dirac measure δ_{x_0} of any set is either 1 or 0, depending on whether x_0 belongs to the set or not. Then in order for the above limit to be 0, for n large enough, the above sets should have δ_{x_0} measure 0 (as the only other option is 1, which is not small). For those sets to have δ_{x_0} measure 0, they should *not* contain x_0, which is to say that the inequality $|f_n(x) - f(x)| \geq \epsilon$ should *not* be true.

We have shown that $f_n \to f$ in the δ_{x_0} measure is equivalent to: for every $\epsilon > 0$, there is N such that if $n \geq N$ then $|f_n(x) - f(x)| < \epsilon$. This is of course the same as

$$f_n(x_0) \to f(x_0) \quad \text{as } \; n \to \infty\,.$$

Let us now investigate the convergence in the L^1 norm, which means that

$$\|f_n - f\|_{L^1} \to 0 \quad \text{as } n \to \infty\,.$$

But in our case, using part b) we have

$$\|f_n - f\|_{L^1} = \int_X |f_n - f| \, d\delta_{x_0} = |f_n - f|\,(x_0) = |f_n(x_0) - f(x_0)|\;,$$

so convergence in the L^1 norm is equivalent to

$$|f_n(x_0) - f(x_0)| \to 0 \quad \text{as } n \to \infty\,.$$

Therefore, in the context of the Dirac measure centered at a point x_0, both convergence in measure and in the L^1 norm are simply equivalent with pointwise convergence exactly at the point x_0, that is, with

$$f_n(x_0) \to f(x_0) \quad \text{as } n \to \infty\,.$$

□