NINTH WEEK HOMEWORK ASSIGNMENT

These problems are about some basic properties (left unproven in class) of the integral of *measurable, non-negative* functions.

Problem 1. Prove that if $f, g \colon \mathbb{R}^d \to [0, \infty]$ are Lebesgue measurable and if $f \le g$ a.e. then $\int f \le \int g$.

Note: Keep in mind that at this point, linearity has not been yet established. So the only thing you could use is the *definition* of the Lebesgue integral of a measurable, non-negative function.

Problem 2. Prove the remaining case from class of the "horizontal truncation" property:

Let $f \colon \mathbb{R}^d \to [0, \infty]$ be a Lebesgue measurable function, and for each integer $n \ge 1$ define its horizontal truncation $f_n(x) := \min\{f(x), n\}$. Prove that if $\int f = +\infty$ then

$$\lim_{n \to \infty} \int f_n = +\infty.$$

Problem 3. Prove the remaining case from class of the "vertical truncation" property:

Let $f \colon \mathbb{R}^d \to [0, \infty]$ be a Lebesgue measurable function, and for each integer $n \ge 1$ define its vertical truncation $f_n := f \cdot \mathbf{1}_{\{x \colon |x| \le n\}}$. Prove that if $\int f < \infty$ then

$$\int f \le \lim_{n \to \infty} \int f_n \, .$$

Note: I should explain what I mean by $|x|$.

If $d = 1$, $|x|$ is just the absolute value of x, so $\{x \colon |x| \le n\} = [-n, n]$.

If $d > 1$, then write $x \in \mathbb{R}^d$ as $x = (x_1, x_2, \dots, x_d)$, and define $|x| := \max\{|x_1|, |x_2|, \dots, |x_d|\}$. So the set $\{x \colon |x| \le n\} = [-n, n]^d$, which is a box in $\mathbb{R}^d$.

Problem 4. Prove that if $f \colon \mathbb{R}^d \to [0, \infty]$ is Lebesgue measurable and if $c \in [0, \infty)$, then

$$\int c\, f = c \int f \, .$$

The goal of the next few problems is to establish the *additivity* of the Lebesgue integral of *non-negative measurable* functions. That is, we will show that if $f, g\colon \mathbb{R}^d \to [0,\infty]$ are Lebesgue measurable, then

$$\int (f+g) = \int f + \int g\,.$$

We will accomplish this goal in a few steps.

Problem 5. Prove first the inequality

$$\int f + \int g \le \int (f+g)\,.$$

Hint: Begin with a simple function *below* f and one below g, add up, use the additivity of the integral for *simple* functions, take supremums.

Problem 6. Assume that f and g "live in a box" (the same box). Prove the reverse inequality under this assumption:

$$\int (f+g) \le \int f + \int g\,.$$

Hint: Begin with a simple function *above* f and one above g, add up, use the additivity of the integral for *simple* functions, take infimums.

Problem 7. Remove the assumption from the previous problem that f and g "live in a box". In other words, prove that for any measurable, nonnegative functions f and g, we have

$$\int (f+g) \le \int f + \int g\,.$$

Hint: Truncate the functions f and g horizontally and vertically; the truncations will live in boxes, so the previous problem will apply to them; to conclude, use a limiting argument (namely the horizontal and vertical truncation properties).

At some point in the argument, it may help to know (prove this separately that if f_n, g_n and $(f+g)_n$ represent horizontal truncations of f, g and $f+g$ respectively, then

$$(f+g)_n \le f_n + g_n.$$