EIGHTH WEEK HOMEWORK ASSIGNMENT

$\diamond$ The next few problems are about some basic properties (left unproven in class) of the integral of *simple, non-negative* functions.

Problem 1. Prove that if $f, g\colon \mathbb{R}^d \to [0, +\infty]$ are simple functions, and if $c \in [0, +\infty]$, then

$$\int_{\mathbb{R}^d} (f+g)(x)\,dx = \int_{\mathbb{R}^d} f(x)\,dx + \int_{\mathbb{R}^d} g(x)\,dx$$

$$\int_{\mathbb{R}^d} c\,f(x)\,dx = c \int_{\mathbb{R}^d} f(x)\,dx.$$

Problem 2. Let $f\colon \mathbb{R}^d \to [0, +\infty]$ be a simple function. Prove that if $f(x) < \infty$ a.e. and if $\operatorname{supp}(f)$ has finite measure then

$$\int_{\mathbb{R}^d} f(x)\,dx < \infty.$$

Problem 3. If $f, g\colon \mathbb{R}^d \to [0, +\infty]$ are simple and if $f = g$ a.e. then $\int f = \int g$.

$\diamond$ The next few problems are about basic properties of the integral of *simple* functions that may take both positive and negative values.

Problem 4. Prove that if $f, g\colon \mathbb{R}^d \to \mathbb{R}$ are simple functions, and if $c \in \mathbb{R}$, then

$$\int_{\mathbb{R}^d} c\,f(x)\,dx = c \int_{\mathbb{R}^d} f(x)\,dx.$$

Problem 5. If $f, g\colon \mathbb{R}^d \to \mathbb{R}$ are simple functions and if $f \leq g$ a.e. then $\int f \leq \int g$.

$\diamond$ The next few problems are about basic properties of *measurable, non-negative* functions.

Problem 6. If $f\colon \mathbb{R}^d \to [0, +\infty]$ is a Lebesgue measurable function and if $\phi\colon [0, \infty] \to [0, \infty]$ is *continuous*, then $\phi \circ f$ is Lebesgue measurable.

Problem 7. If $f, g\colon \mathbb{R}^d \to [0, +\infty]$ are Lebesgue measurable and if $c \in [0, \infty)$, then $f + g$, $c\,f$ and $f \cdot g$ are Lebesgue measurable.

Problem 8. Prove that $f\colon \mathbb{R}^d \to [0, \infty]$ is a simple function if and only if it is measurable and it only takes finitely many values.